Diploma 6th SEM

CONTROL SYSTEM & COMPONENTS

By

Ms. Kalyani Behera

(Lecture)

Department of Electronics Communication Engineering

VIKASH POLYTECHNIC, BARGARH

Transfer function.

The transfer function of a control system is defined as the ratio of laplace transform of output variable to Lapace transform of input variable taking initial condition zero.

$$r(t) \longrightarrow \boxed{g(t)} \longrightarrow c(t)$$

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$$

$$G(s) = \frac{L\,c(t)}{L\,r(t)} = \frac{C(s)}{R(s)}\Bigg|_{\text{initial condition}=0}$$

Poles & Zeros of a Transfer Function

$$G(s) = \frac{C(s)}{R(s)} = \frac{a_0(s-s_0)+a_1(s-s_1)+\cdots+a_n(s-s_n)}{b_0(s-s_a)+b_1(s-s_b)+\cdots+b_m(s-s_m)}$$

$$= \frac{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}$$

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$$= \frac{k(s-s_1)(s-s_2) \cdots (s-s_n)}{(s-s_a)(s-s_b) \cdots (s-s_m)}$$

* If $s = s_1$ & $s = s_2$ the transform function becomes zero so these are called zeros to that transfer function.

* If $s = s_a$ and $s = s_b$ the transfer function becomes infinite so these are called poles to that transfer function.

The points for which the transfer function become infinite are called poles to that transfer function.

Block diagram reduction technique

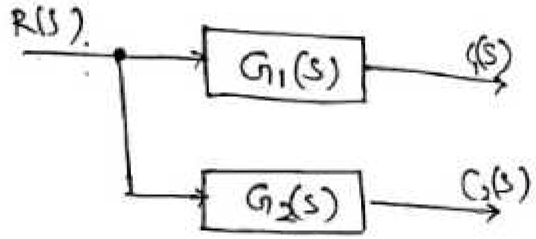
* It is not convenient to derive a complete transfer function for a complex control system.

* The transfer function of each element of a control system is represented by a block diagram.

* The symbolic representation in a short form gives a pictorial representation relating to output and input of a control system based on cause and effect.

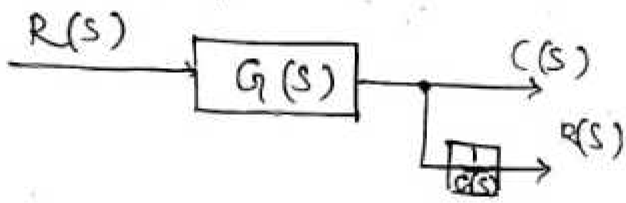
$$G(s) = \frac{C(s)}{R(s)}$$

Take off point :



* Application of one input source to two or more system is represented by take off point.

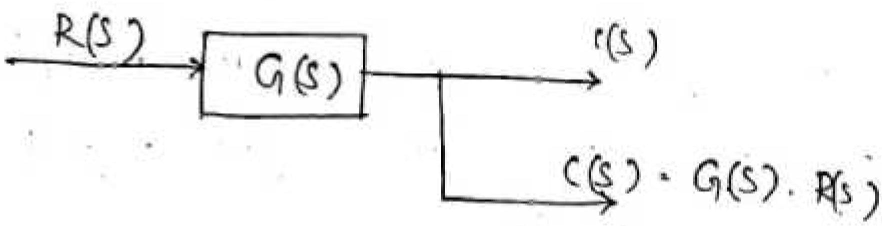
* Shifting of take off point before a block to after a block.

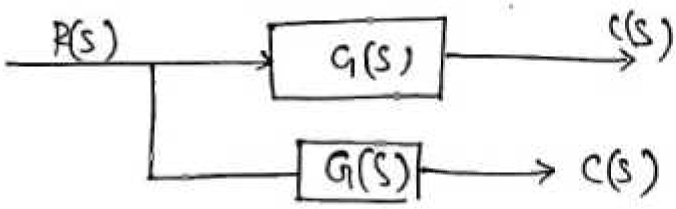


$$G(s) = \frac{C(s)}{R(s)}$$

$$\Rightarrow R(s) = \frac{C(s)}{G(s)}$$

* Shifting a take off pointe after a block to before a block.





$C(s) = G(s)\, R(s)$

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* Blocks in Cascade

$$G_1(s) = \frac{C_1(s)}{R(s)}$$

$$G_2(s) = \frac{C_2(s)}{C_1(s)}$$

$$G_3(s) = \frac{C(s)}{C_2(s)}$$

$$G_1(s)\ G_2(s)\ G_3(s)$$

$$= \frac{C_1(s)}{R(s)} \cdot \frac{C_2(s)}{C_1(s)} \cdot \frac{C(s)}{C_2(s)}$$

$$= \frac{C(s)}{R(s)}$$

$$G(s) = \boxed{\frac{C(s)}{R(s)} = G_1(s) \cdot G_2(s) \cdot G_3(s)}$$

* Summing point

Summing point Represent sumission of two or more input signal intering into a system.

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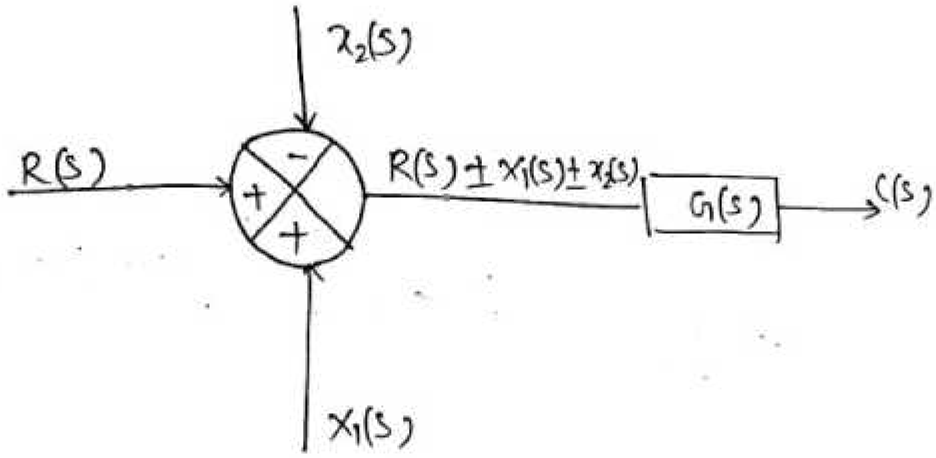
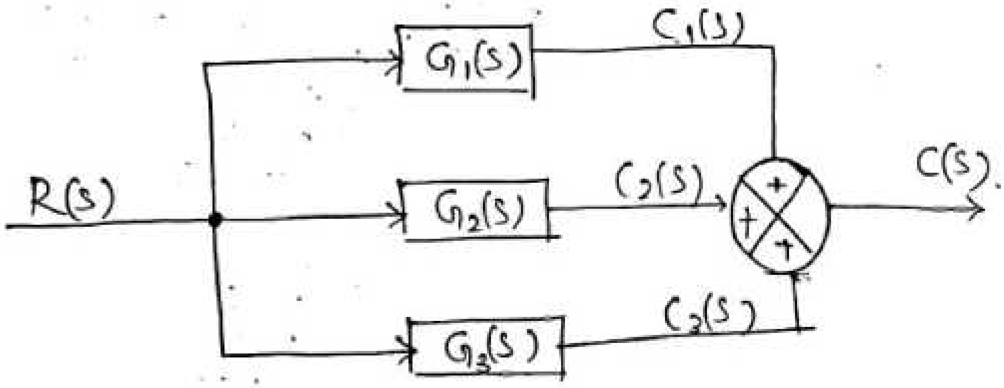


* **Blocks in parallel**



$C_1 = R(s) G_1(s)$

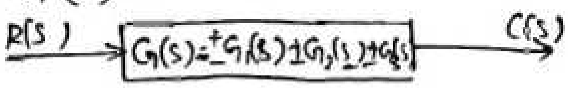
$C_2 = R(s) G_2(s)$

$C_3 = R(s) G_3(s)$

$$C(s) = \pm C_1(s) \pm C_2(s) \pm C_3(s)$$
$$= \pm R(s) G_1(s) \pm R(s) G_2(s) \pm R(s) G_3(s)$$
$$= \pm R(s) \left(G_1(s) \pm G_2(s) \pm G_3(s) \right)$$

$$\Rightarrow \frac{C(s)}{R(s)} = \pm G_1(s) \pm G_2(s) \pm G_3(s)$$

$$\Rightarrow G(s) = \pm G_1(s) \pm G_2(s) \pm G_3(s).$$



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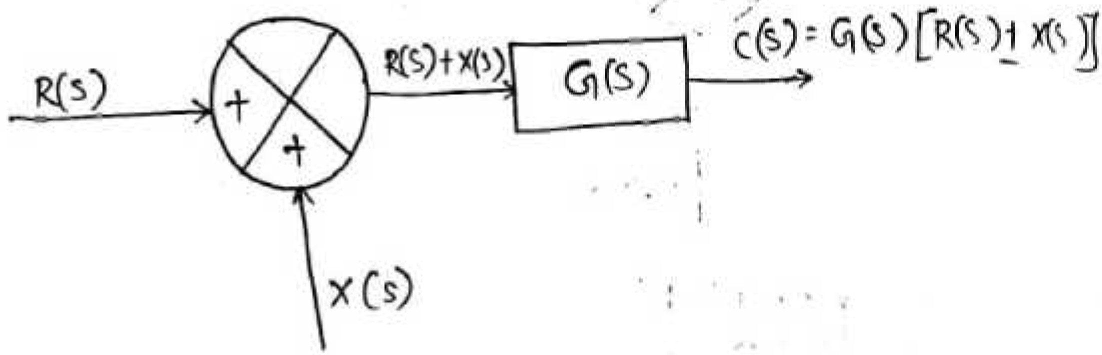
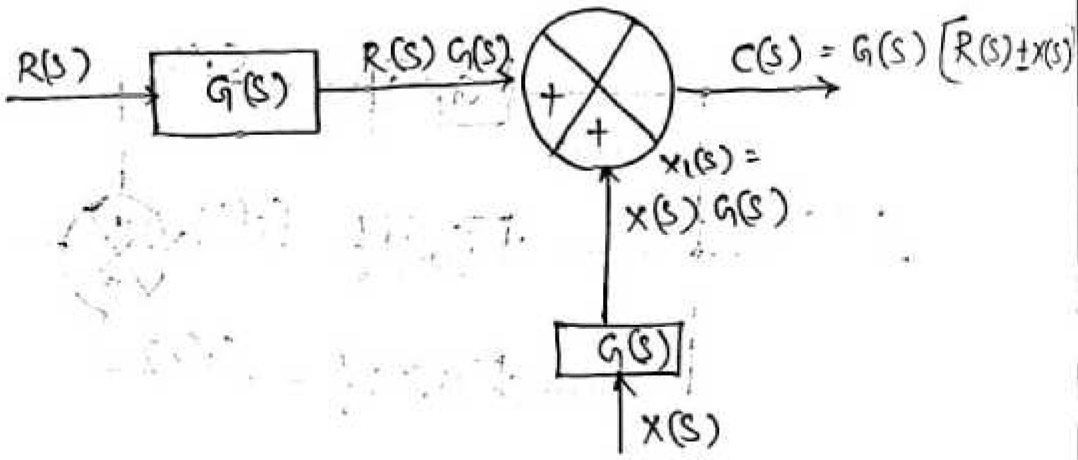
*Shifting of a summing point before a block to after a block





$X(S) = R(S)\, G(S)$

$C(S) = [R(S) \cdot G(S)] \pm X_1(S)$

$\Rightarrow X_1(S) + [R(S)\; G(S)] = G(S)\,[R(S) \pm X(S)]$

$= G(S)\, R(S) \pm G(S)\, X(S)$

$\Rightarrow X_1(S) = \cancel{G(S) R(S)} \pm G(S)\, X(S) \pm (-)\cancel{R(S) G(S)}$

$= G(S) \cdot X(S)$

to get $X_1(S) = G(S)\, X(S)$ put a extra Block $G(S)$

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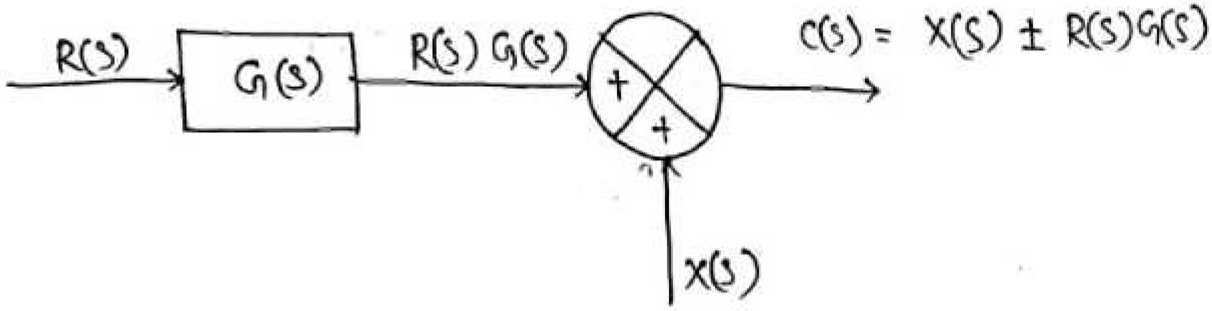
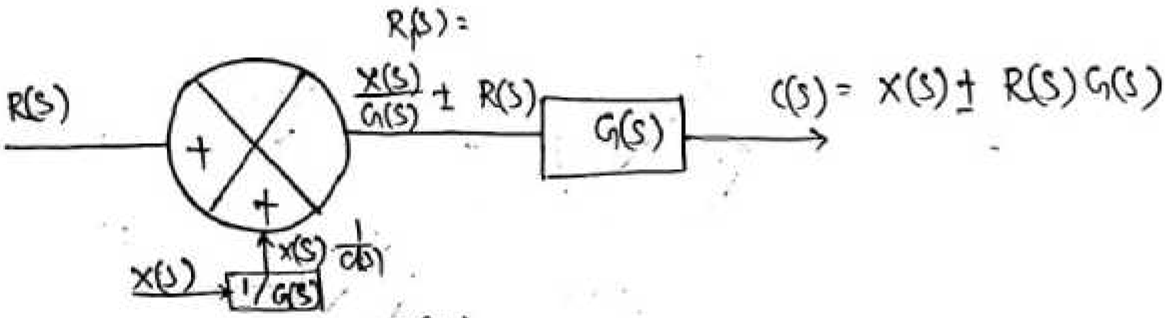
* Shifting a summing point after a block to before a block





$$R_1(S) = \frac{C(S)}{G(S)}$$

$$= \frac{X(S) \pm R(S)\,G(S)}{G(S)}$$

$$= \frac{X(S)}{G(S)} \pm R(S)$$

But $\frac{X(S)}{G(S)} \pm R(S) = R(S) \pm X_1(S)$

$$\Rightarrow X_1(S) = \frac{X(S)}{G(S)}.$$

to get $X_1(S) = \frac{X(S)}{G(S)}$ put a extra block $\frac{1}{G(S)}$

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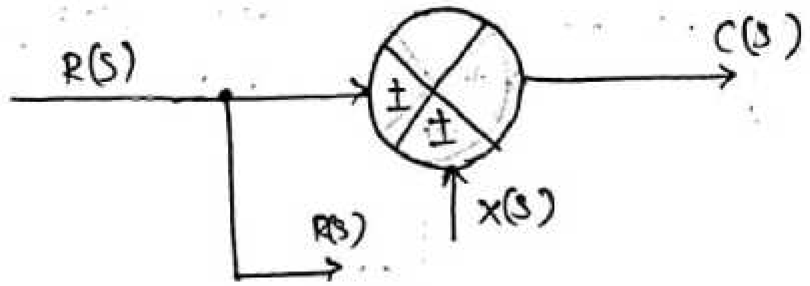
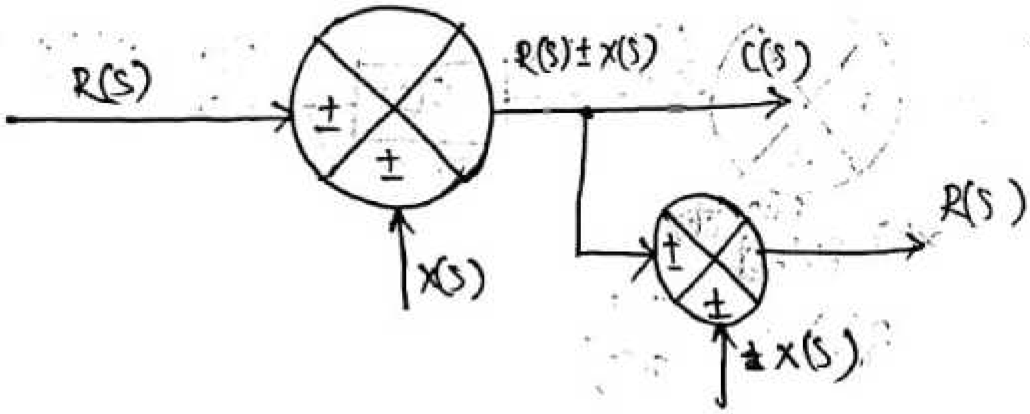
* Shifting a take off point before a summing point to after a summing point





$$C(S) = R(S) \pm X(S)$$

But $\quad R(S) = C(S) \pm X_1(S)$

$$\Rightarrow X_1(S) = R(S) - C(S)$$

$$= R(S) - [R(S) \pm X(S)]$$

$$= R(S) - R(S) \pm X(S)$$

$$= X(S)$$

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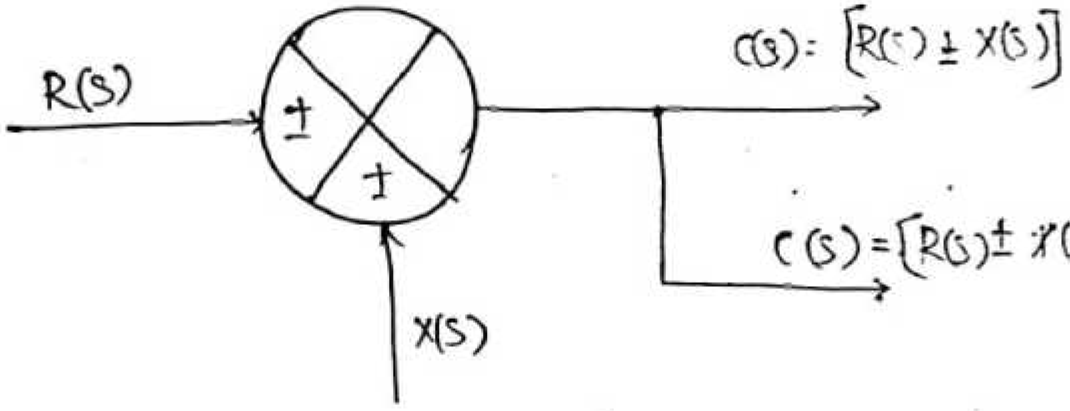
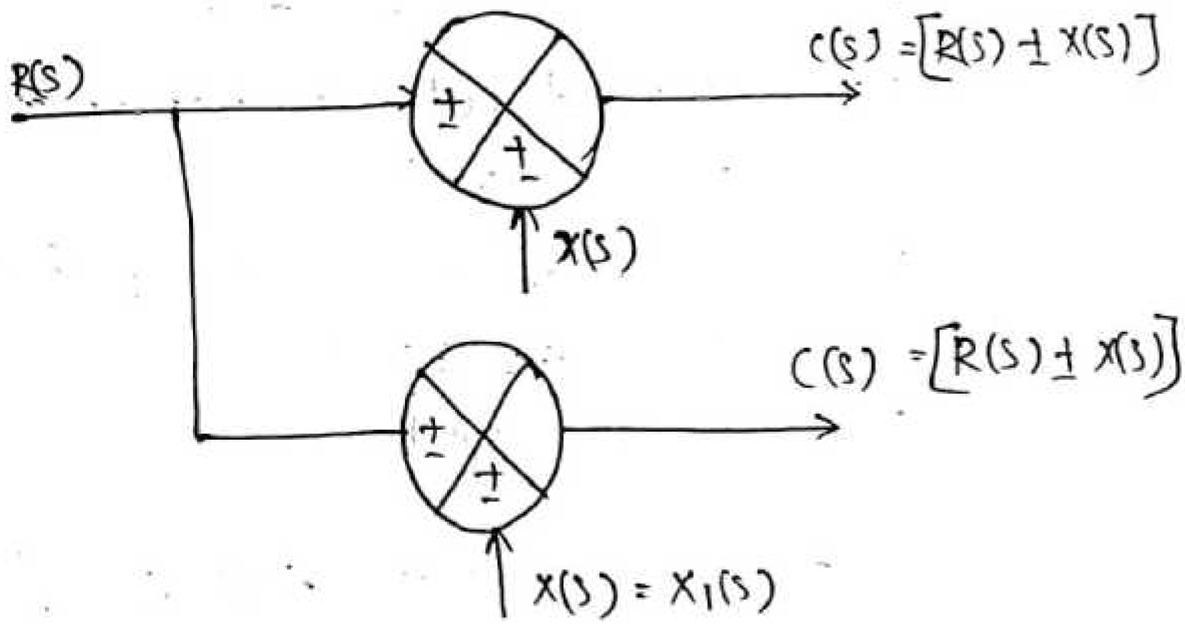
* shifting a take off point after a summing point to before a summing point.





$$C(S) = R(S) \pm X_1(S)$$

$$\Rightarrow X_1(S) = C(S) - R(S)$$

$$= \cancel{R(S)} \pm X(S) - \cancel{R(S)}$$

$$= \pm X(S)$$

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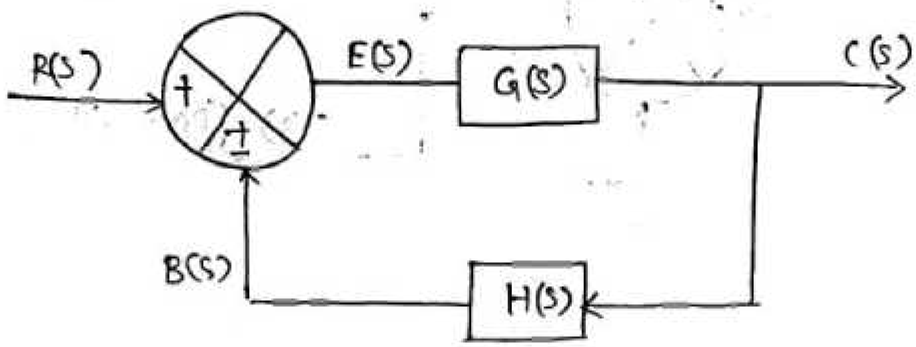
*Elimination of Summing point in a close loop transfer function.



R(s) = Reference i/p signal

G(s) = Forward path transfer function

C(s) = Output signal

H(s) = Feedback path transfer function

B(s) = Feedback signal

E(s) = Error signal.

$C(s) = E(s) \cdot G(s) \Rightarrow G(s) = \frac{C(s)}{E(s)}$

& $B(s) = C(s) \cdot H(s)$

But $E(s) = R(s) \pm B(s)$

$$E(s) = \frac{C(s)}{G(s)}$$

$$\Rightarrow \frac{C(s)}{G(s)} = R(s) \pm B(s)$$

$$= R(s) \pm C(s) \cdot H(s)$$

$$\Rightarrow \frac{C(s)}{G(s)} \pm C(s) \cdot H(s) = R(s)$$

$$\Rightarrow C(s)\left(\frac{1}{G(s)} \pm H(s)\right) = R(s)$$

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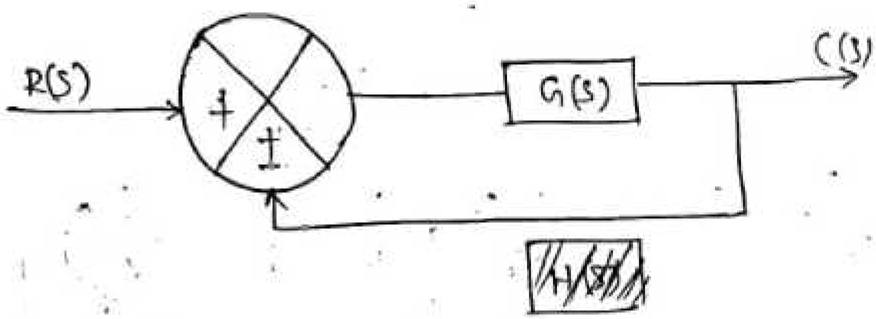
$$\Rightarrow \frac{R(S)}{C(S)} = \left(\frac{1}{G(S)} \pm H(S)\right)$$

$$= \frac{1 \pm G(S).H(S)}{G(S)}$$

$$\boxed{\Rightarrow \frac{C(S)}{R(S)} = \frac{G(S)}{1 \pm G(S).H(S)}}$$

In case of unity feedback path transfor function.



$$\frac{C(s)}{R(S)} = \frac{G(S)}{1 \pm G(S)}.$$

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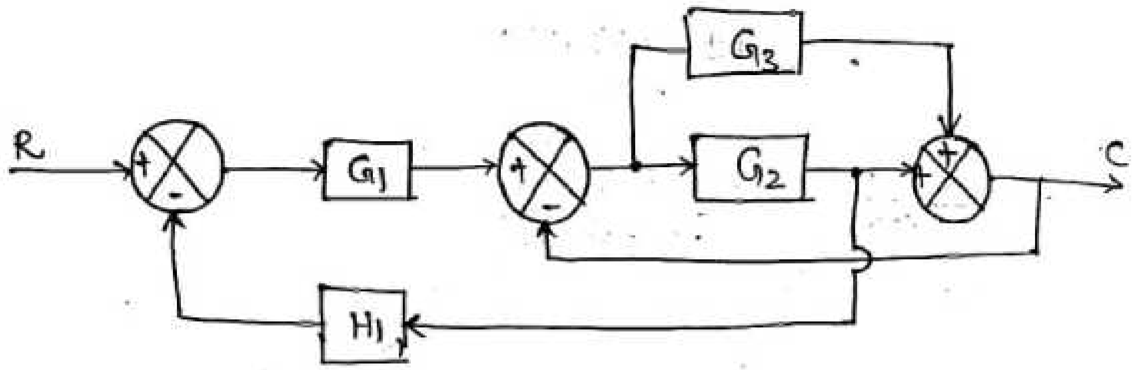
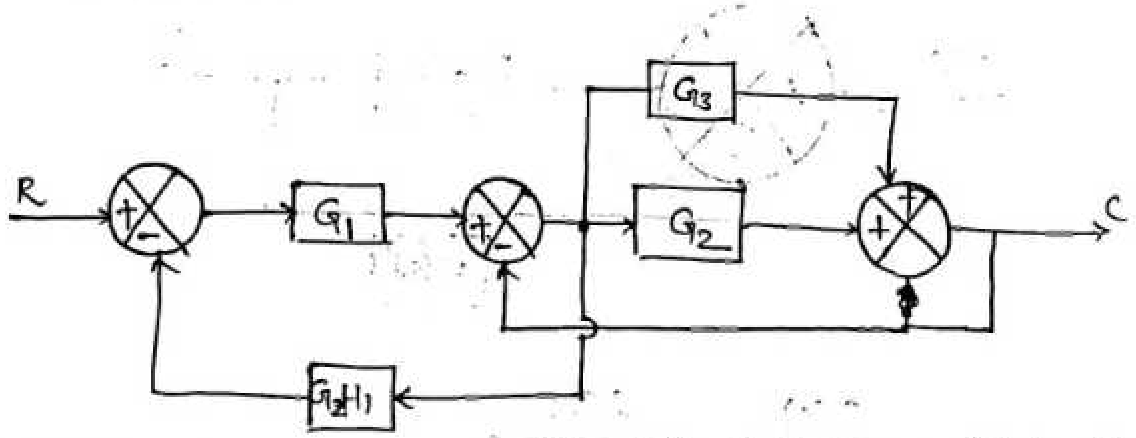
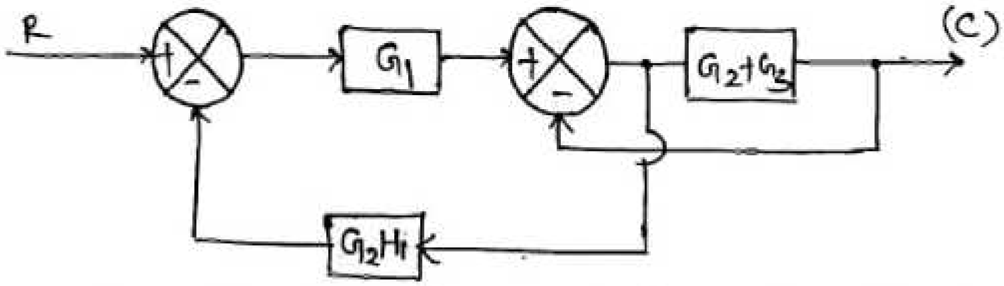
Prob



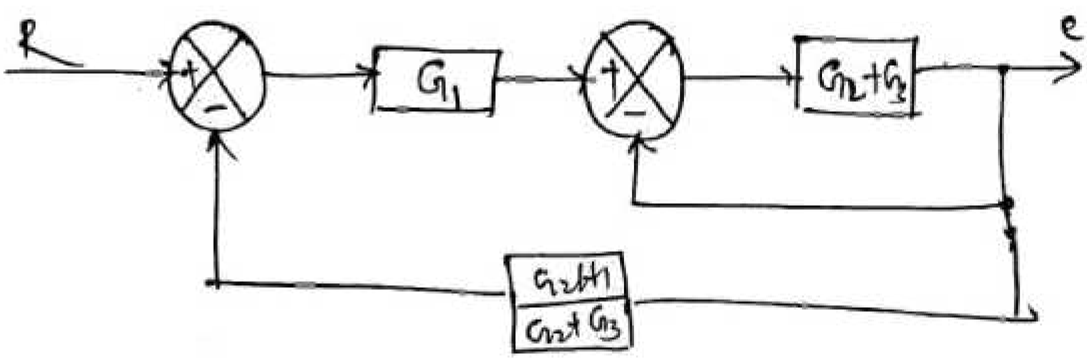
1) shifting the takeup point after the block G_2 to before the block G_2



2) Eleminating summing point after the ~~point~~ block G_2



3) shifting the take off point before the block G_2+G_3 to after the block G_2+G_3



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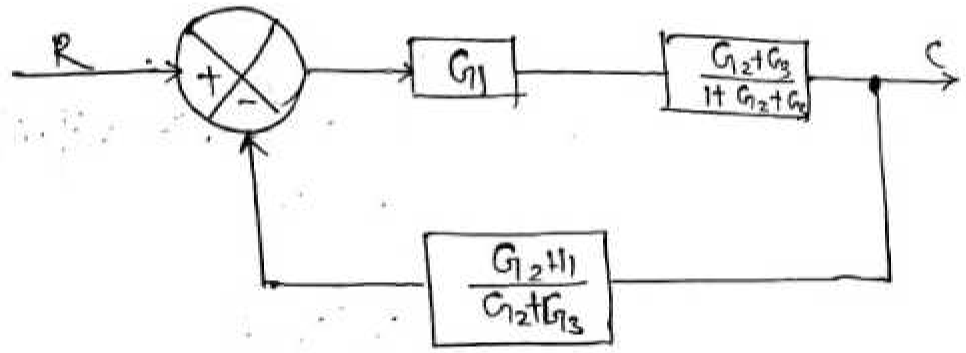
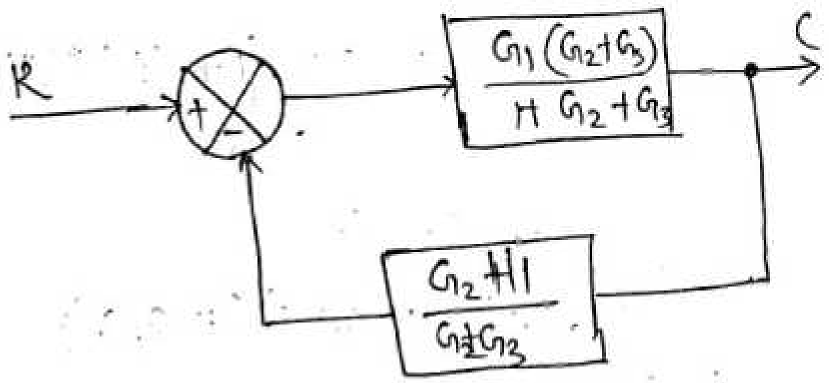
4) Eliminating the summing point before the block $G_2 + G_3$



5) Combining the block G_1 & $\frac{G_2+G_3}{1+G_2+G_3}$ connected in cascade.



6) Eleminating the summing point and solving the close loop transfer function.

$$\frac{C}{R} = \frac{\dfrac{G_1(G_2+G_3)}{1+G_2+G_3}}{1+\dfrac{G_1(G_2+G_3)}{1+G_2+G_3} \times \dfrac{G_2 H_1}{G_2+G_3}}$$

$$= \frac{\dfrac{G_1(G_2+G_3)}{1+G_2+G_3}}{1+\dfrac{H_1 G_1 G_2 (G_2+G_3)}{G_2+G_3(1+G_2+G_3)}}$$

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$$= \frac{\dfrac{G_1(G_2+G_3)}{1+G_2+G_3}}{1+\dfrac{H_1G_1G_2(G_2+G_3)}{G_2+G_3(1+G_2+G_3)}}$$

$$= \frac{\dfrac{G_1(G_2+G_3)}{1+G_2+G_3}}{\dfrac{G_2+G_3(1+G_2+G_3)+H_1G_1G_2(G_2+G_3)}{G_2+G_3(1+G_2+G_2)}}$$

$$= \frac{\dfrac{G_1(G_2+G_3)}{1+G_2+G_3}}{\dfrac{\cancel{G_2+G_3}(1+G_2+G_3+H_1G_1G_2)}{\cancel{G_2+G_3}(1+\cancel{G_2+G_3})}}$$

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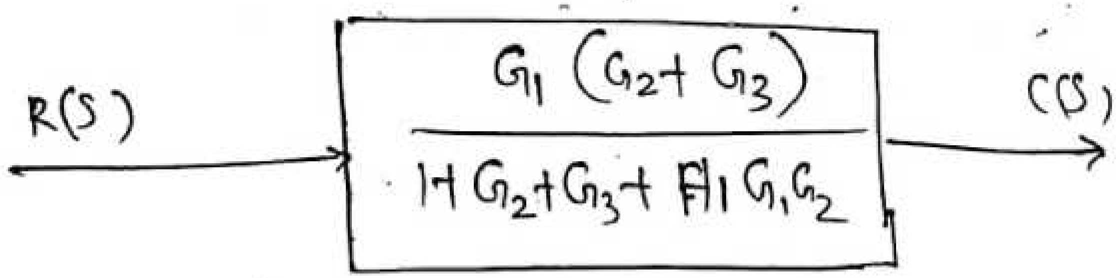
$$= \frac{G_1(G_2 + G_3)}{1 + G_2 + G_3 + H_1 G_1 G_2}$$



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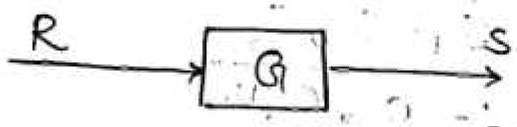
CHAPTER-II

Signal Flow Graph

* Block diagram gives a picturial representation of control system in short form of transfer function.
* To get the overall transfer function of a control system by block diagram reduction technique. it is very time consuming and tridious.
* Another way of representation of a control system by eliminating summing point, take off point and block is called signal flow graph.
* In signal flow graph the variables are represented by point called 'nodes' and the transfer function is called "transmitlance". which is represented by a branch the through which signal can flow.



$S = RG$

$x_2 = ax_1$

x_1 = input variable nodes
x_2 = output " "

* The arrow head represent the direction of signal flow.

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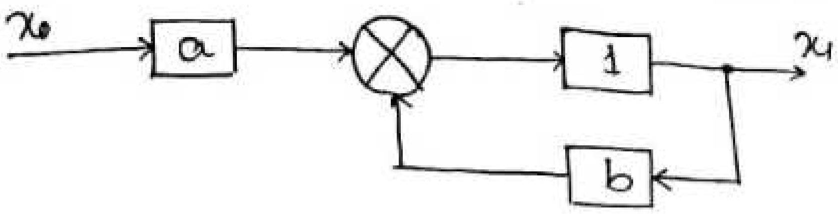

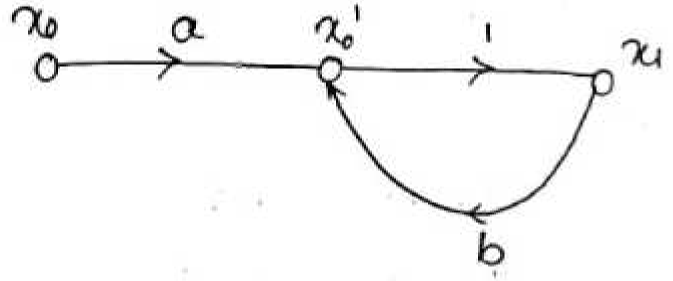

$$x_0' = ax_0 + x_0'b$$

$$x_1 = x_0' \times 1$$

$$x_1 = x_0' = ax_0 + x_0'b$$

Here x_0 is the input verable node.
x_1 is the output " "
a = transmissance between the x_0 & x_0' which is called forward ~~bo~~ path. transmittance.
b = loop transmittance.

Rules to draw signal flow graph

* The signal travel along a brance in the direction of an arrow.
* The input signal multiplied with the transmittor to get the output signal.
* The input signal at a node is the sum of all signal entering to that node.

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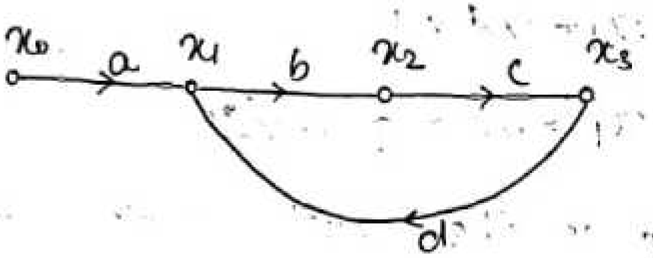
* A node transmit signal in all branches leaving to that node.

$$x_1 = a x_0$$

$$x_2 = b x_1 = b \times a x_0 = ab x_0$$

$$x_3 = c x_2 = c \times ab x_0 = abc x_0$$



$$x_1 = a x_0 + d x_3$$

$$x_2 = b x_1$$

$$x_3 = c x_2 = c(b x_1) = bc x_1$$

$$x_1 = a x_0 + d(bc x_1)$$

$$\Rightarrow x_1 - bcd x_1 = a x_0$$

$$\Rightarrow x_1(1 - bcd) = a x_0$$

$$\Rightarrow x_1 = \frac{a x_0}{1 - bcd}.$$

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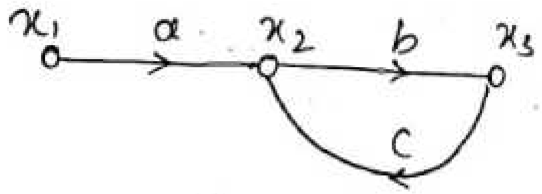
$$x_3 = bcx_1$$
$$= bc\left(\frac{ax_0}{1-bcd}\right)$$

$$\boxed{T.F = \frac{x_3}{x_0} = \frac{abc}{1-bcd}.}$$

$p = abc$ = Forward path transmittance
$L = bcd$ - loop transmittance.

Prob



$$x_2 = ax_1 + cx_3$$
$$x_3 = x_2 b.$$
$$= b(ax_1 + cx_3)$$
$$\Rightarrow x_3 = abx_1 + bcx_3$$
$$\Rightarrow x_3 - bcx_3 = abx_1$$
$$\Rightarrow (1-bc)x_3 = abx_1$$
$$\boxed{\Rightarrow \frac{x_3}{x_1} = \frac{ab}{1-bc}} \quad T.F$$

$p = ab$ = Forward path transmittance
$L = bc$ = Loop transmittance

Forward Path :

It is a path of a signal flow graph starting from input variable to output variable without ~~ero~~ touching a node more than 1.

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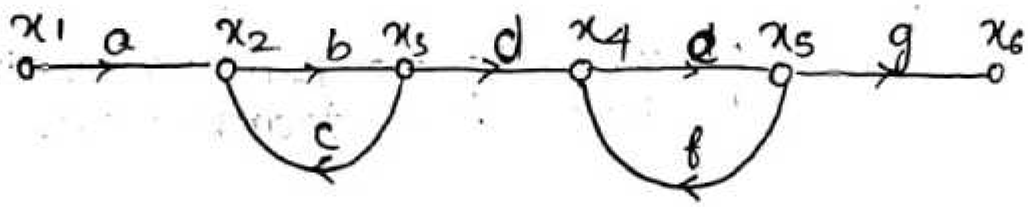
Loop

It is a path in a signal flow graph where starting and end node is same..



$$x_2 = a x_1 + c x_3 \text{ —— (i)}$$

$$x_3 = b x_2 \text{ —— (ii)}$$

$$x_4 = d x_3 + f x_5 \text{ —— (iii)}$$

$$x_5 = e x_4 \text{ —— (iv)}$$

$$x_6 = g \cdot x_5 \text{ —— (v)}$$

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x_4 :-

$$TF = \frac{abdge}{1 - bc - ef + bcef}$$

forward path transmitance = P

loop transmitance $L_1 = bc$

" " $L_2 = ef$

$$TF = \frac{P}{1 - (L_1 + L_2) + L_1 L_2}$$

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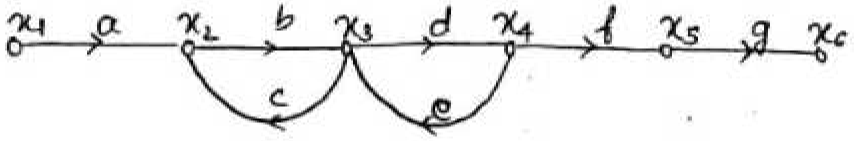
Ex



$x_2 = ax_1 + cx_3$ — (i)

$x_3 = bx_2 + ex_4$ — (ii)

$x_4 = dx_3$ — (iii)

$x_5 = fx_4$ — (iv)

$x_6 = gx_5$ — (v)

$P =$ Forward path transmittance $= abdfg$

$L_1 =$ loop transmittance

$= bc$

$L_2 =$ loop transmittance

$= de.$

$x_6 = gx_5$

$= gfx_4$

$= gfdx_3$

$= dgfx_3$

$x_3 = bx_2 + ex_4$

$= bx_2 + dex_3$

$= x_3(b - de$

$\Rightarrow x_3 - dex_3 = bx_2$

$\Rightarrow x_3(1-de) = bx_2$

$\Rightarrow x_3 = \left(\frac{b}{1-de}\right)x_2$

$x_2 = ax_1 + cx_3$

$\Rightarrow ax_1 = x_2 - cx_3$

$= \frac{b}{1-de} \quad x_2 - c\frac{b}{1-de}x_2$

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$$= x_2\left(1-\frac{bc}{1-de}\right)$$

$$\Rightarrow x_2 = \frac{a x_1}{1-\frac{bc}{1-de}}$$

$$x_6 = dgf\left(\frac{b}{1-de}\right)x_2$$

$$= dgf\left(\frac{b}{1-de}\right)\left(\frac{a x_1}{1-\frac{bc}{1-de}}\right)$$

$$= \frac{abdgf\,x_1}{(1-de)\left(1-\frac{bc}{1-de}\right)}$$

$$= \frac{abdgf\,x_1}{\cancel{1-de}\left(\frac{1-de-bc}{\cancel{1-de}}\right)}$$

$$= \frac{abdgf\,x_1}{1-de-bc}$$

$$\Rightarrow \frac{x_6}{x_1} = \frac{abdgf}{1-de-bc}$$

$$\Rightarrow T = \frac{P}{1-L_1-L_2} = \frac{P}{1-(L_1+L_2)}$$

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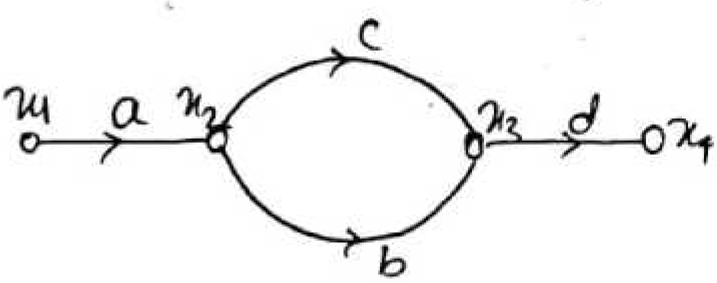

$x_2 = a x_1$

$x_3 = b x_2 + c x_2 = (b+c) x_2$

$x_4 = d x_3$

$x_4 = d(b+c) x_2$

$= d(b+c) a x_1$

$x_4 = (abd + adc) x_1$

$$\frac{x_4}{x_1} = abd + adc = P_1 + P_2$$

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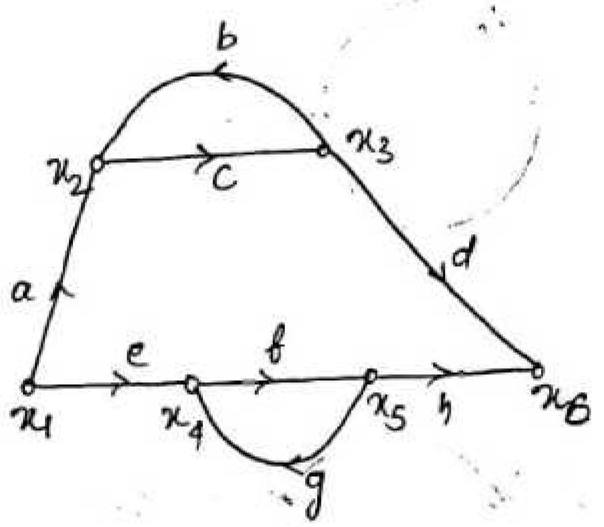

$$x_2 = ax_1 + bx_3$$

$$x_3 = cx_3$$

$$x_4 = ex_1 + gx_5$$

$$x_5 = fx_4$$

$$x_6 = hx_5 + dx_3$$

$$\frac{x_6}{x_1} = T = \frac{acd(1-fg) + efh\,(1-bc)}{1-(bc+fg) + bcfg.}$$

$P_1 = acd \qquad L_1 = bc$

$P_2 = efg. \qquad L_2 = fg.$

$$= \frac{P_1\,(1-L_2) + P_2\,(1-L_1)}{1-(L_1+L_2) + L_1L_2}$$

$$= \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$T = \frac{\sum_{k=1}^{3} P_k\Delta k}{\Delta}$$

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$$T = \frac{\sum_{k=1} P_k \Delta_k}{\Delta}$$

P_k : forward path transmittance of k_{th} path.

Δ_k = Path factor of the k^{th} forward path.

Δ = Gain Determinant.

* Mason's gave a formula to calculate the overall transmittance of a signal flow graph is given by. $T = \frac{\sum_{k=1}^{n} P_k \Delta_k}{\Delta}$ is called Mason's Gain formula.

Δ = 1 - (sum of all possible loop gain) + (sum of product of pair of non touching groups loop) - (sum of product of triple non-touching loop)

* Δ_k is the path factor of k^{th} path is part of della is it is obtained by removing the loop gain which are touching the k^{th} forward path from 'Δ'.

$$T = \sum_{k=1} \frac{P_k \Delta_k}{\Delta}$$

$P_1 = acd \qquad P_2 = efh.$

$\Delta = 1 - (L_1 + L_2) + L_1 L_2$

$\Delta_1 = 1 - L_1$

$\Delta_2 = 1 - L_2$

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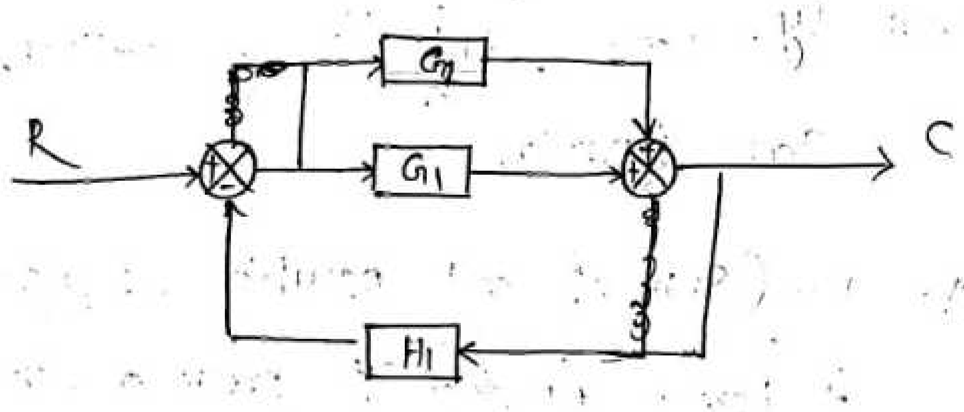
$$T = \frac{\sum_{k=1} P_k \Delta_k}{\Delta}$$

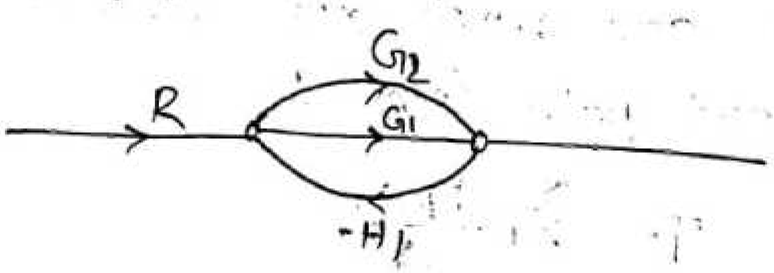
$$= \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{acd\,(1-L_2) + efh\,(1-L_1)}{1-(L_1+L_2) + L_1L_2}$$

<u>Prob</u>



$$\frac{C}{R} = \frac{G_1+G_2}{1+(G_1+G_2)H_1} \Rightarrow \frac{G_1+G_2}{1+G_1H_1+G_2H_2}$$



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$$T = \frac{\sum P_k \Delta_k}{\Delta}$$

$$= \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$P_1 = G_1$, $P_2 = G_2$

$L_1 = -G_1H_1$ $\quad L_2 = -G_2H_1$

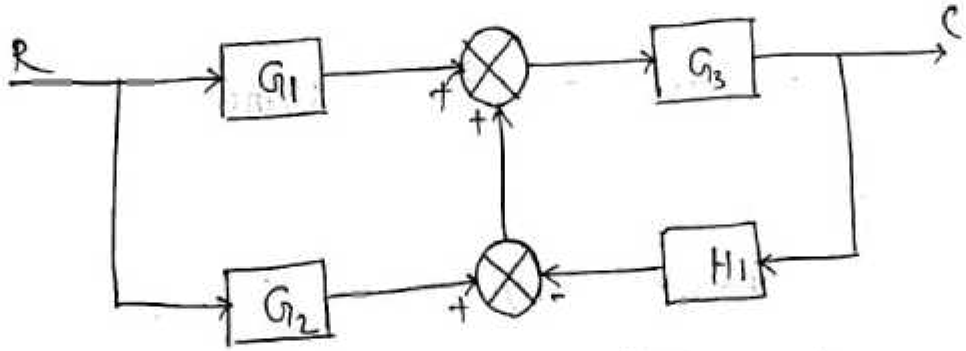
$\Delta = 1 - (L_1 + L_2)$

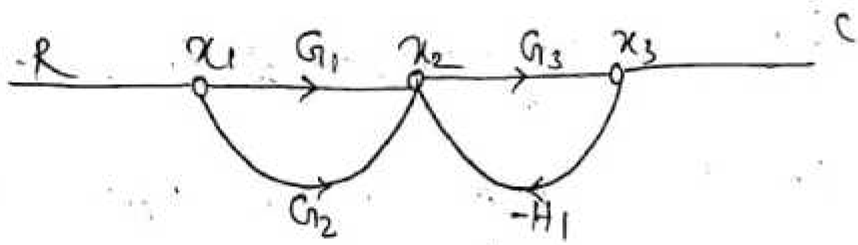
$\Delta_1 = 1$ $\quad \Delta_2 = 1$

$$= \frac{G_1 \times 1 + G_2 \times 1}{1 - (-G_1H_1 - G_2H_1)}$$

$$= \frac{G_1 + G_2}{1 + G_1H_1 + G_2H_1}$$

Ex.





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$$x_2 = G_1 x_1 + G_2 x_1 - H_1 x_3$$

$$x_3 = G_3 x_2$$

$$= G_3 \left(G_1 x_1 + G_2 x_1 - H_1 x_3 \right)$$

$$= G_3 x_1 (G_1 + G_2) - G_3 H_1 x_3$$

$$= (G_3 G_1 + G_3 G_2) x_1 - G_3 H_1 x_3$$

$$\Rightarrow x_3 + G_3 H_1 x_3 = (G_3 G_1 + G_3 G_2) x_1.$$

$$\Rightarrow x_3 (1 + G_3 H_1) = (G_3 G_1 + G_3 G_2) \cdot x_1.$$

$$\Rightarrow \frac{C}{R} = \frac{x_3}{x_1} = \frac{G_3 G_1 + G_3 G_2}{1 + G_3 H_1}$$

$$T = \frac{\sum_K P_k \Delta_k}{\Delta}$$

$P_1 = G_1 G_3$ = Transmittance of first forward path.

$P_2 = G_2 G_3$: Transmittance of 2^{nd} forward path

$L = -G_3 H_1$ = Loop transmittance.

$$\Delta = 1 - L$$

$$= 1 - (-G_3 H_1)$$

$= 1 + G_3 H_1$ = Gain Determinant

$\Delta_1 = 1$ = Path factor of first forward path

$\Delta_2 = 1$ = Path " " 2nd " path.

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$$T = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

$$= \frac{G_1 G_3 + G_2 G_3}{1 + G_3 H_1}$$

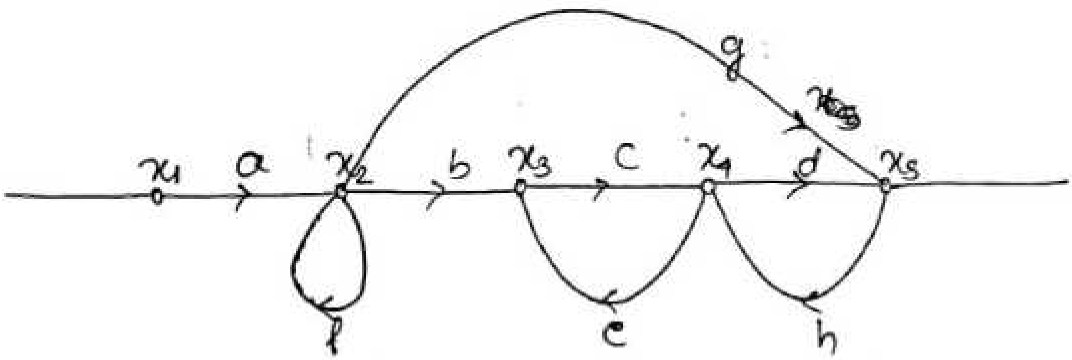
<u>Ex</u> :

$x_2 = a x_1 + f x_2$

$x_3 = b x_2 + e x_4$

$x_4 = c x_3 + h x_5$

$x_5 = d x_4 + g x_2$



$$T = \frac{\sum P_k \Delta_k}{\Delta}$$

P_1 = Forward path transmittance of 1st forward path. = $abcd$.

P_2 = Forward path transmittance of 2nd forward path = ag.

$L_1 = f$ = loop Gain

$L_2 = ce$

$L_3 = dh$.

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$\Delta = 1 - (L_1 + L_2 + L_3) + (L_1 L_2 + L_1 L_3)$

$= 1 - (f + ce + dh) + (cef + fdh)$

$= 1 - f - ce - dh + cef + fdh.$

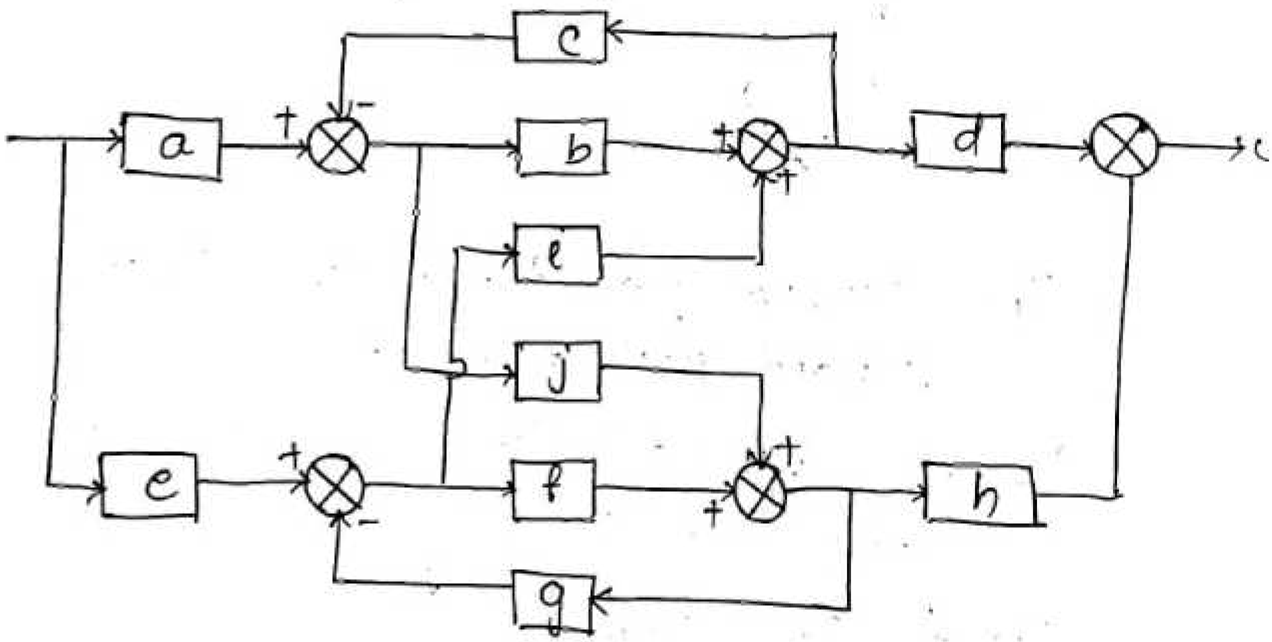
Δ_1 = Path factor of first forward path $= 1$

Δ_2 = Path factor of 2^{nd} " " $= 1 - L_2$

$= 1 - ce$

$$T = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{abcd + ag(1-ce)}{1 - f - ce - dh + fce + fdh}$$



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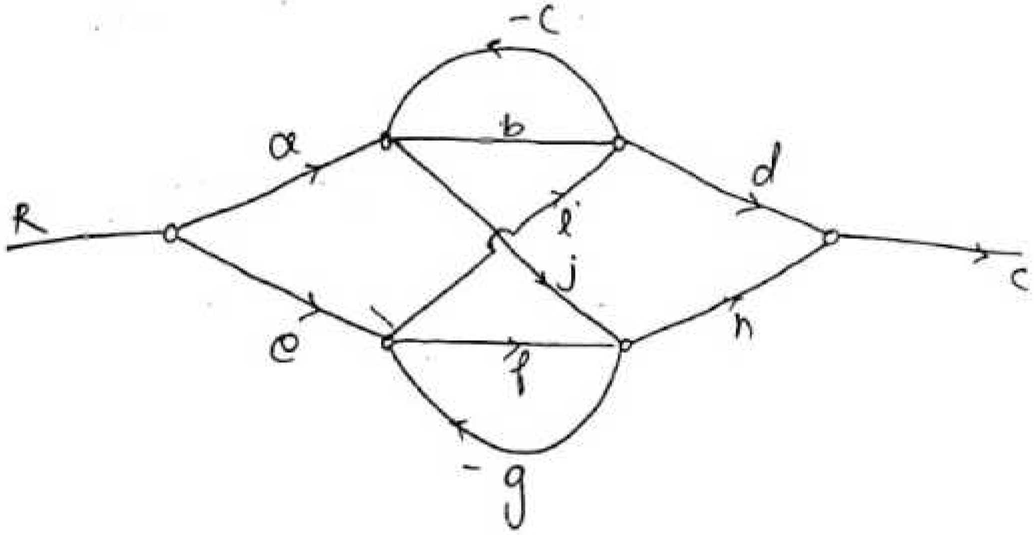

$$T = \frac{P_k \Delta_k}{\Delta}$$

$P_1 = abd$

$P_2 = efh.$

$P_3 = ajh$

$P_4 = eld$

$P_5 = aj(-g)ld.$

$P_6 = el(-c)jh.$

$L_1 = -bc \quad l_2 = -fg \quad l_3 = j(-g)l(-c)$

$= jglc$

$\Delta = 1 - (L_1 + l_2 + l_3) + L_1 L_2$

$= 1 - (-bc + -fg + jglc) + bcfg$

$= 1 + bc + fg - jglc + bcfg.$

$\Delta_1 = 1 - L_2 \qquad \Delta_2 = 1 - L_1$

$= 1 + fg \qquad = 1 + bc$

$\Delta_3 = 1 \quad , \Delta_4 = 1 \qquad \Delta_5 = 1 \quad , \Delta_6 = 1 ,$

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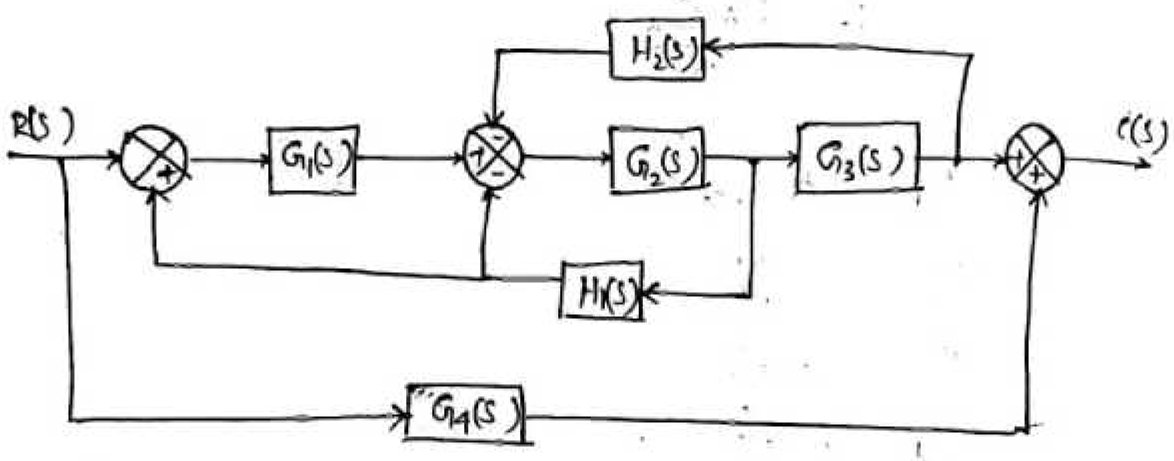
$$T = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6}{\Delta}$$

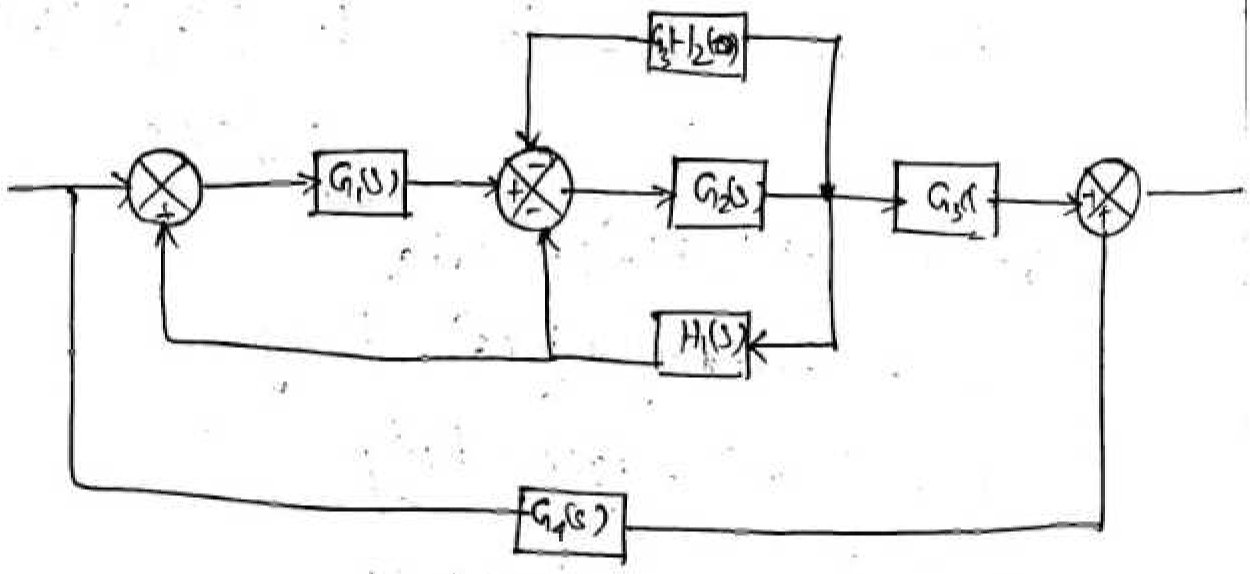
$$= \frac{abd(1+fg) + efh(1+bc) + ajh + eid - ajgid + ecjh}{1 + bc + fg - jgec + bcfg}$$

Prob



Step-1

Shifting the take off point ~~before~~ after the block G_3 to ~~after~~ before the block G_3



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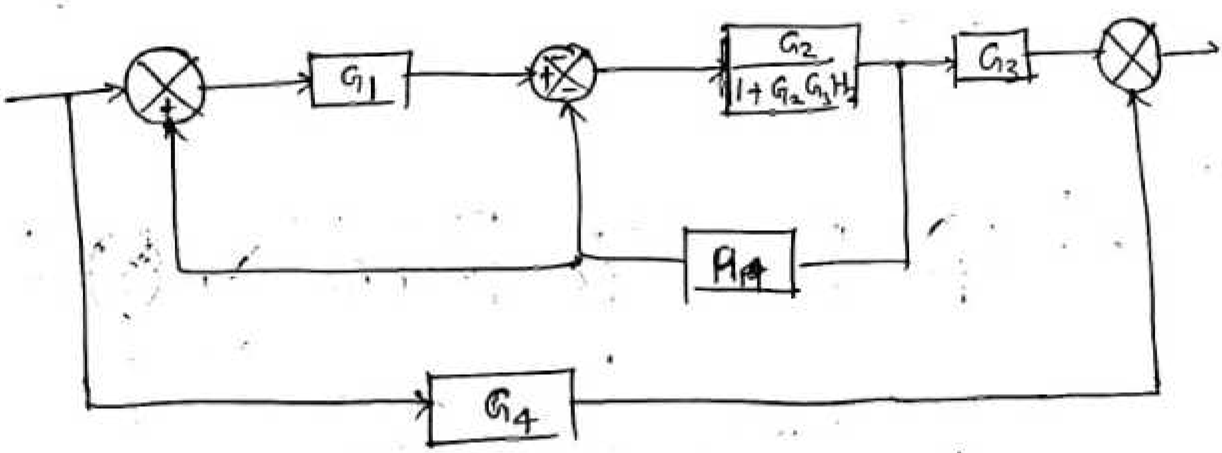
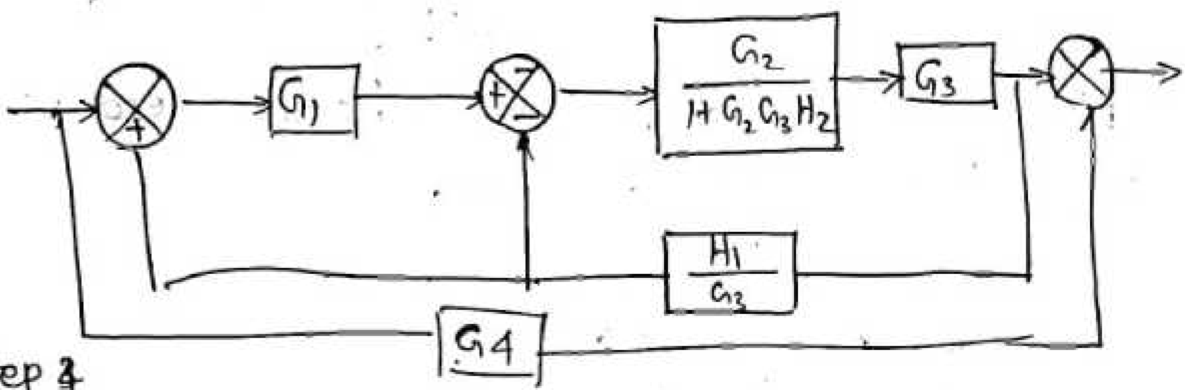
Step · 2

Eleminating the close loop having feedback path transfer function $G_3 H_2$



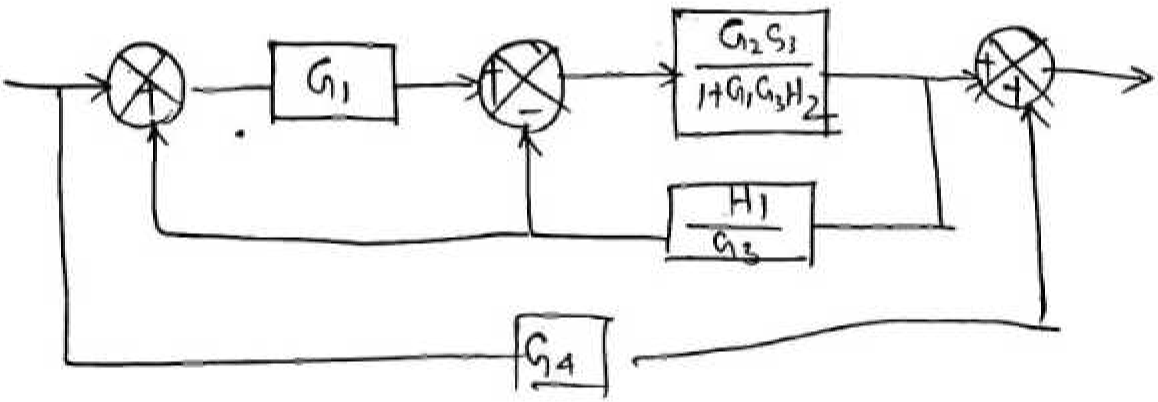
Step 3

shifting the takeoff point before G_3 to after G_3



Step 4

Combining the block connected in cascade.



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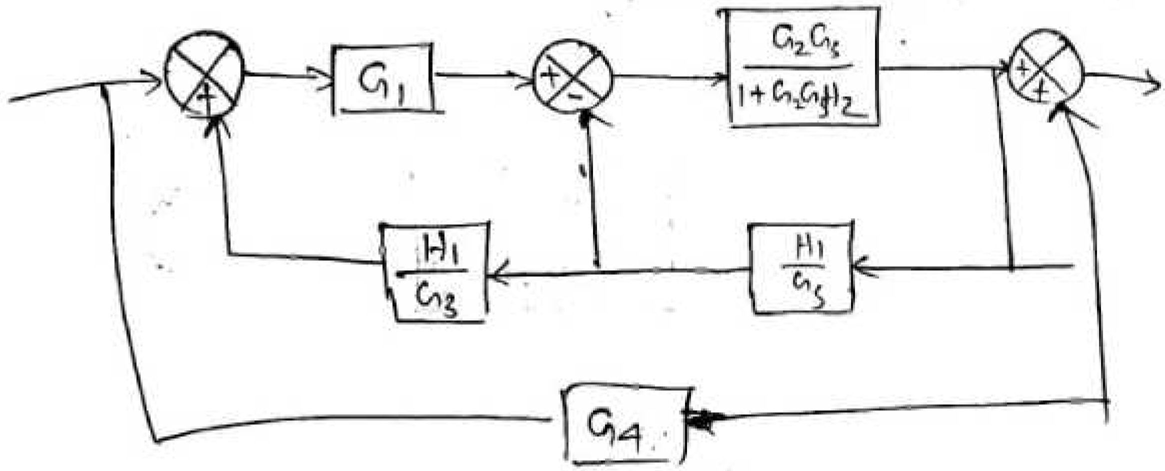


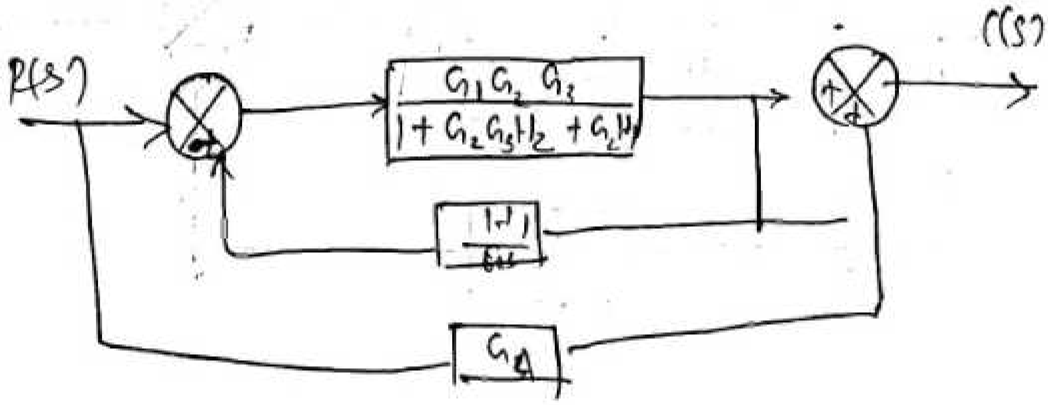
Eleminatting the close loop transfer function having the (+ve) feedback $\left(\frac{H_1}{G_3}\right)$

$$\frac{G_2 G_3}{1+G_2G_3H_2} = X$$

$$\frac{H_1}{G_3} = Y$$

$$\frac{X}{1+XY} = \frac{\frac{G_2G_3}{1+G_2G_3H_2}}{1+\frac{G_2\cancel{G_3}}{1+G_2G_3H_2}\times\frac{H_1}{\cancel{G_3}}}$$

$$= \frac{G_2G_3}{1+G_2G_3H_2+G_2H_1}$$



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Solving the close loop having the feedback both transfer function. $\frac{H_1}{G_3}$

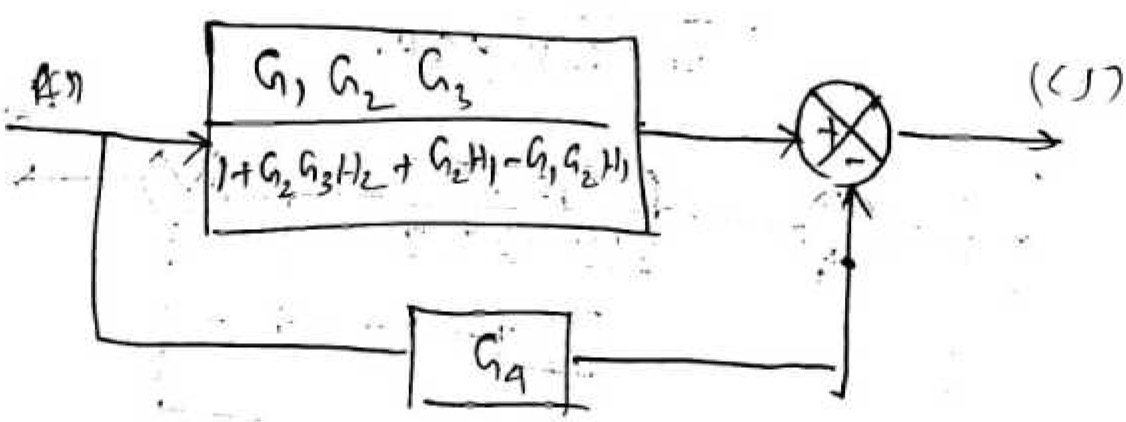
$$\boxed{\frac{G_1G_2G_3}{1+G_2G_3H_2+G_2H_1}} \quad (1)$$

$$\frac{X}{1-XY}$$

$$\Rightarrow \frac{\frac{G_1G_2G_3}{1+G_2G_3H_2+G_2H_1}}{1-\frac{G_1G_2\cancel{G_3}}{1+G_2G_3H_2+G_2H_1}\times\frac{H_1}{\cancel{G_3}}}$$

$$\therefore \frac{\frac{G_1G_2G_3}{1+G_2G_3H_2+G_2H_1}}{\frac{1+G_2G_3H_2+G_2H_1-G_1G_2H_1}{1+G_2G_3H_2+G_2H_1}}$$

$$= \frac{G_1G_2G_3}{1+G_2G_3H_2+G_2H_1-G_1G_2H_1}$$



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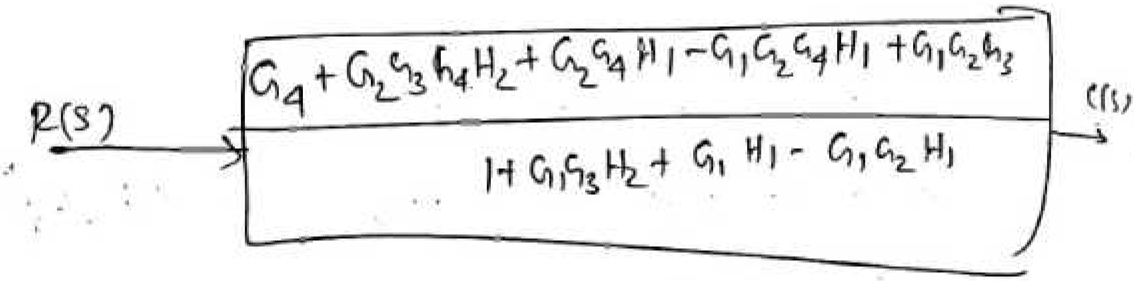
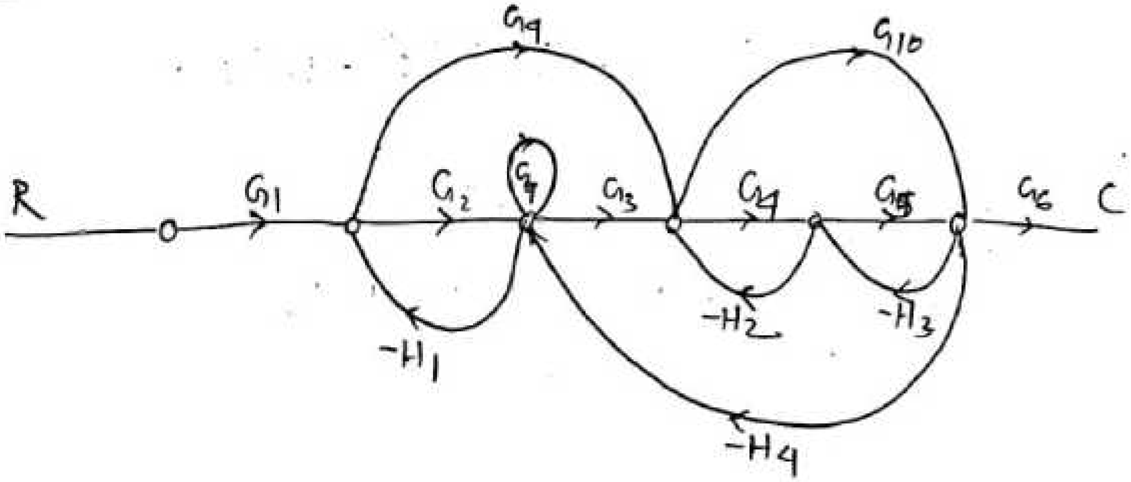
solving the parallel path

$$G_4 + \frac{G_1 G_2 G_3}{1 + G_2 G_3 H_2 + G_2 H_1 - G_1 G_2 H_1}$$

$$= \frac{G_4 + G_2 G_3 G_4 H_2 + G_2 G_4 H_1 - G_1 G_2 G_4 H_1 + G_1 G_2 G_3}{1 + G_1 G_3 H_2 + G_1 H_1 - G_1 G_2 H_1}$$



Prob



Forward path

$P_1 = G_1 G_2 G_3 G_4 G_5 G_6$

$P_2 = G_1 G_9 G_{10} G_6$

$P_3 = G_1 G_9 G_4 G_5 G_6$

$P_4 = G_1 G_2 G_3 G_{10} G_6$

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Loop

$L_1 = -G_2 H_1$

$L_2 = G_7$

$L_3 = -G_3 G_4 G_5 H_4$

$L_4 = -G_3 G_{10} H_4$

$L_5 = -G_4 H_2$

$L_6 = G_8$

$L_7 = -G_5 H_3$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_6 + L_7) + (L_1 L_5 + L_1 L_6 + L_1 L_7 + L_1 L_5 + L_2 L_6 + L_2 L_7 + L_3 L_6 + L_4 L_6)$$

$\Delta_1 = 1 \qquad \Delta_2 = 1 - (L_2 + L_6) + L_2 L_6$

$\Delta_3 = 1 - L_2 \qquad \Delta_4 = 1 - L_6$

$$T = \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3 + P_4 \Delta_4}{\Delta}$$

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Time Response Analysis

* Time response analysis means how a system behaves with respect to time for a specified input signal.

* The input to a control system can't be assesed before hand. therefore input test signals are applied.
* The initial part of the time response of a control system is called <u>transient state</u>.
* The post transient period the steady state is achived.

Steady State:

Theoritically the steady state means a state of the output of a control system as the time approches infinity. after initiation of input

Transient Response

* The response of the output of a control system after input signal is given is called transient response.
* The transient part of the time response shows the nature of response i.e (Oscillatory or overdamped)
* It also indicate the speed of the system.

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Steady State Response

* After transient response ie when $t \to \infty$ steady state response is achived the steady state part of the time response gives the accuracy of the a control system.

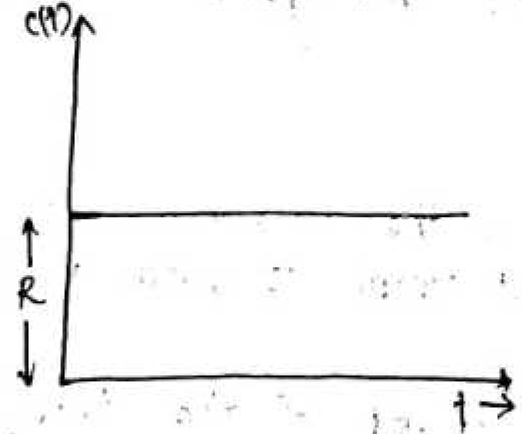
* In this part wee will find the steady state error.

Input Test Signal

Specified input test signal applied for time response analysis of a control system are given below.

Step function.

* Step function is describe sudden application of input signals.



$r(t) = R \quad t \geq 0$

$r(t) = 0 \quad t < 0$

* If R = 1 unit the step function is called unit step function.

* The step function is also called displacement function.

$$\mathcal{L}\, r(t) = \mathcal{L}(1) = \frac{1}{s}$$

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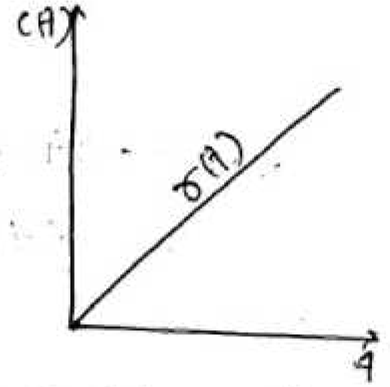
Ramp - Function

Ramp function is a gradual application of input signal with respect to time.



$r(t) = Rt$ for $t \geq 0$

$= 0$ for $t < 0$

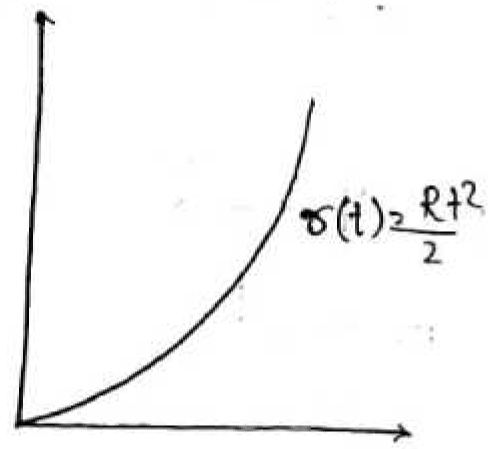
* If $R = 1$ then $r(t) = t$ is called unit ramp function.

$$\mathcal{L}\, r(t) = \mathcal{L}(t) = \frac{1}{s^2}$$

* Ramp function is also called velocity function.

Parabolic function

Parabolic function is describe more gradual application of ~~input~~ input in comparison with ramp function.



$r(t) = \frac{Rt^2}{2}$ $t \geq 0$

$= 0$ $t < 0$

* If $R = 1$ then $r(t) = \frac{t^2}{2}$ is called unit parabolic function

$$\mathcal{L}\, r(t) = \mathcal{L}\left(\frac{t^2}{2}\right) = \frac{1}{s^3}$$

* Parabolic function is called ~~displacement~~ acceleration function.

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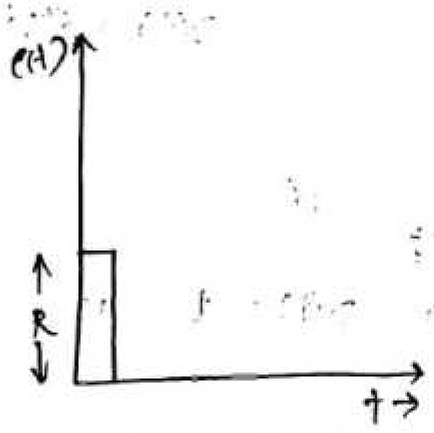
Impulse function.

Input is suddenly applied as a shock for a very short duration of time.



$r(t) = R\,\delta(t)$ $t = 0$

$= 0$ $t \neq 0$

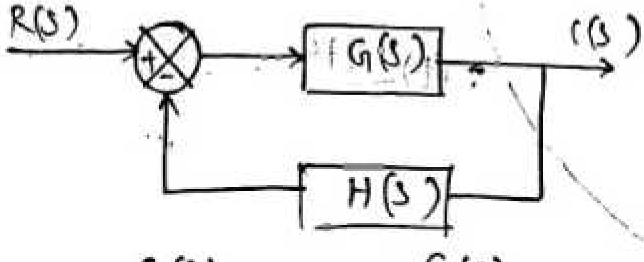
* If $R = 1$ then $r(t) = \delta(t)$ the function is called unit impulse function.

$$\text{Impulse function} = \frac{d}{dt}(\text{step function})$$

$$= \frac{d}{dt} r(t) = \frac{d}{dt}(1)$$

Applying laplace transfer function.

$$= S\,R(S) = S \times \frac{1}{S} = 1$$



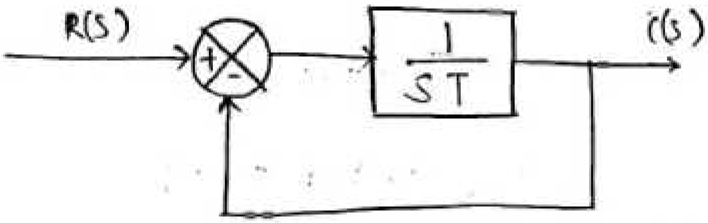
$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S).H(S)}$$

* If the denominator of the overall transfer function is equated to zero is called characteristic equation.

* The heighest power of 's' denominator of overall transfer function of a control system is called order of the control system.
* If heighest power of 's' in the denominator is '1' is called first order control system.
* If highest power of 's' in the denominator is '2' is called 2^{nd} order control system.

Time response of a first order control system

1) Appling Unit step input.



$$G(s) = \frac{1}{ST}$$

$$H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\frac{1}{ST}}{1+\frac{1}{ST}\times 1}$$

$$= \frac{\frac{1}{ST}}{\frac{ST+1}{ST}} = \frac{1}{ST+1}$$

$$\frac{C(s)}{R(s)} = \frac{1}{ST+1}$$

$$\Rightarrow C(s) = R(s) \times \frac{1}{ST+1}$$

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$r(t) = 1$

$R(s) = \frac{1}{s}$

$$\Rightarrow C(s) = \frac{1}{s} \times \frac{1}{(sT+1)}$$

$$= \frac{1}{s(sT+1)}$$

Applying partial fraction

$$C(s) = \frac{A}{s} + \frac{B}{sT+1}$$

$$\frac{1}{s(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\frac{1}{s(sT+1)} = \frac{A(sT+1) + B(s)}{s(sT+1)}$$

$$\Rightarrow A + (AT + B)s = 1$$

$$= 0 \cdot s$$

Equating coefficient of 's' and constant term on both side of the eqn

$A = 1$ $\qquad$ $AT + B = 0$

$\Rightarrow T + B = 0$

$\Rightarrow B = -T$

$$\text{so } \frac{1}{s(s+sT)} = \frac{1}{s} + \frac{-T}{1+sT}$$

$$= \frac{1}{s} + -\frac{T}{1+sT}$$

$$= \frac{1}{s} - \frac{T}{1+sT}$$

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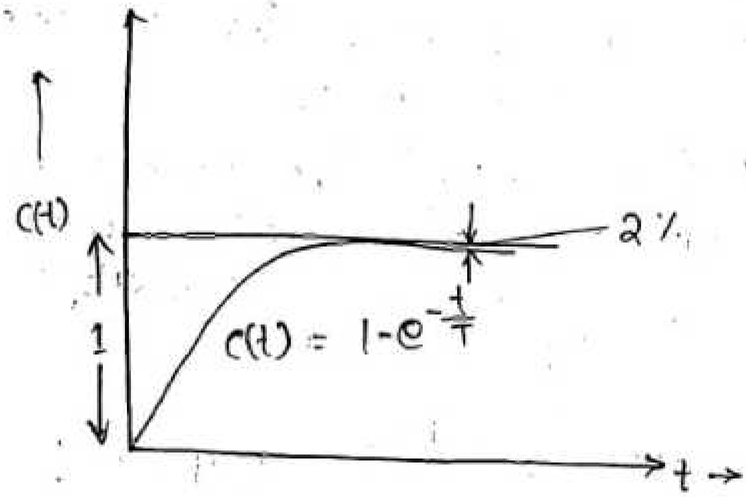
$$C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Taking inverse laplace on the both side.

$$\mathcal{L}^{-1} C(s) = \mathcal{L}^{-1}\frac{1}{s} - \mathcal{L}^{-1}\frac{1}{s+\frac{1}{T}}$$

$$\boxed{C(t) = 1 - e^{-\frac{t}{T}}} \qquad \left(\because \mathcal{L}^{-1}\frac{1}{s+a} = e^{-at} \right.$$



* When the output is within the 2% of the refference input the steady state is achived.

Actual output $= c(t) = 1 - e^{-t/T}$

Refference input $= r(t) = 1$

error $= r(t) - c(t)$

$= 1 - (1 - e^{-t/T})$

$$\boxed{e(t) = e^{-t/T}}$$

Steady state error $= e_{ss} = \lim_{t\to\infty} e(t)$

$= \lim_{t\to\infty} e^{-t/T}$

$= 0$

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Time constant

$$C(t) = 1 - e^{-t/T}$$

one time constant

$$t = T$$

$$C(t) = 1 - e^{-T/T} = 1 - e^{-1} = 0.632$$

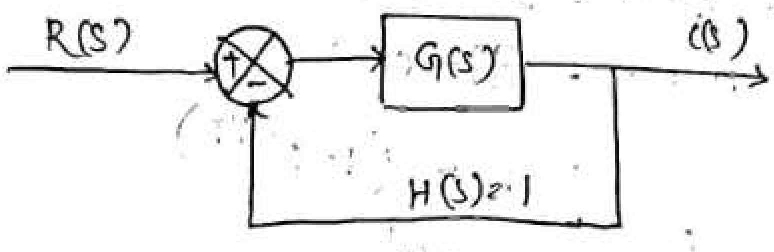
Time constan

* After one time constant the response reaches 63.2 % of the desired value.
* After 4 time constant the response reaches to the desired value. after that steady state is achived. that means 4 time constant is the demarcation between transient state & the steady state.

$$t = 4T$$

$$C(t) = 1 - e^{-4T/T} = 0.99 \simeq 1$$

II) Applying impulse input.



$G(S) = \frac{1}{ST}$

$H(S) = 1$

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S) \cdot H(S)}$$

$$= \frac{1}{S}$$

$$= \frac{1}{1 + ST}$$

$$\Rightarrow C(S) = R(S)\,\frac{1}{1 + ST}$$

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$$C(s) = \frac{1}{1+sT}$$

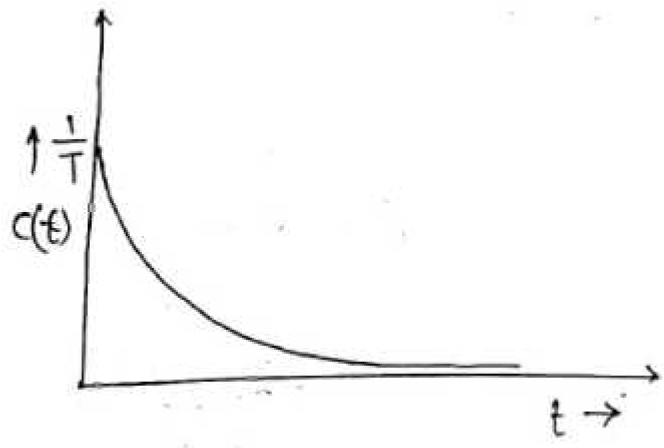
$$= \cancel{\frac{1}{1+sT}}$$

$$= \frac{1}{T\left(\frac{1}{T}+s\right)}$$

$$= \frac{\frac{1}{T}}{s+\frac{1}{T}}$$

Taking inverse laplace transfer functn

$$c(t) = \frac{1}{T}e^{-t/T} \qquad \mathcal{L}^{-1}\frac{1}{s+a} = e^{-at}$$



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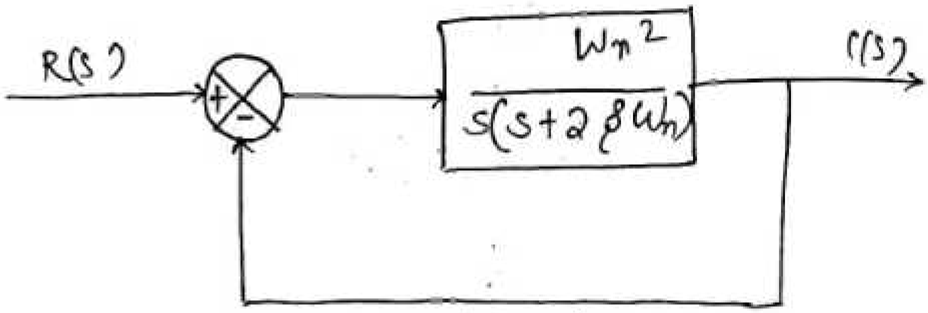
Time Response of a second order Control System



Consider a second order control system ~~unit~~ with unit feedback.

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s).H(s)}$$

$$= \frac{\frac{\omega_n^2}{s(s+2\zeta\omega_n)}}{1+\frac{\omega_n^2}{s(s+2\zeta\omega_n)}}$$

$$= \frac{\omega_n^2}{s(s+2\zeta\omega_n)+\omega_n^2}$$

$$= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\Rightarrow C(s) = R(s)\,\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

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When unit step input is given :-

$$r(t) = 1 \qquad R(s) = \frac{1}{s}$$

$$C(s) = R(s)\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\Rightarrow \qquad = \frac{A(s^2+2\xi\omega_n s+\omega_n^2)+(Bs+C)s}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$= \frac{(A+B)s^2+(2\xi\omega_n A+C)s+A\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

Comparing coefficient of s^2, s & constant term on both side.

$$0.s^2+0.s+\omega_n^2 \quad (A+B)s^2+(2\xi\omega_n A+C)s+A\omega_n^2$$

$$A+B=0 \qquad 2\xi\omega_n A+C=0 \qquad A=1$$

$$B=-1 \qquad 2\xi\omega_n.1+C=0$$

$$C=-2\xi\omega_n$$

$$C(s) = \frac{1}{s} + \frac{-s-2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

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$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2s.\xi\omega_n+(\xi\omega_n)^2-(\xi\omega_n)^2+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2-\xi^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$

put $\omega_n^2(1-\xi^2) = \omega_d^2$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

ω_n = Natural frequency.

ω_d = Damped frequency.

$$c(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$c(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d} \times \frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

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Taking inverse laplace transfer of on both side.

$$= \mathcal{L}^{-1}\frac{1}{s} - \mathcal{L}^{-1}\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\times\mathcal{L}^{-1}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\times e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\times\sin\omega_d t\right)$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right)$$

put $\sqrt{1-\zeta^2} = \sin\phi$ $\quad \zeta = \cos\phi$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\phi.\cos\omega_d t + \cos\phi\sin\omega_d t\right)$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$

let
$\cos\phi = \zeta$
$\sin\phi = \sqrt{1-\zeta^2}$

$$\tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

ω_n = Natural frequency

ζ = Damping ratio

$\zeta = 0 \quad \zeta < 1 \quad \zeta = 1 \quad \zeta > 1$

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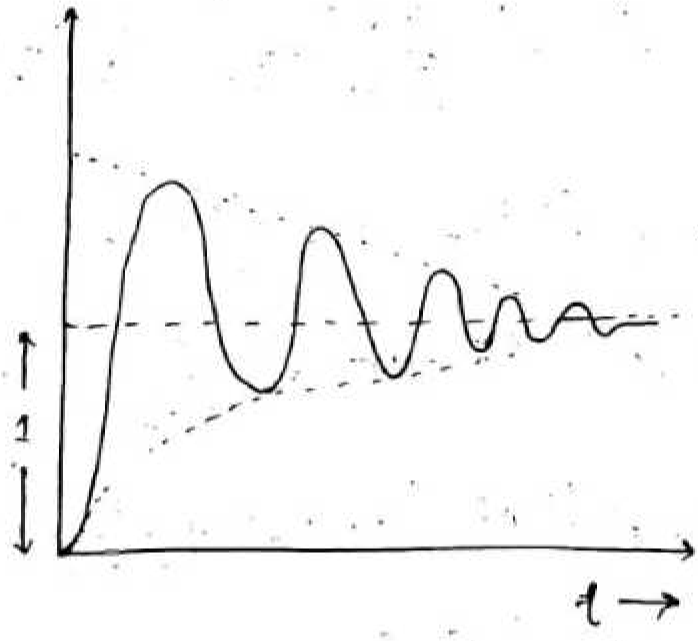

* From this expression $c(t)$ the output is oscillating and the ampletute is decreasing if $\xi < 1$. the system is called under damped system.

<u>When $\xi = 0$</u> :

$$C(t) = 1 - \frac{e^{\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_n \sqrt{1-\xi^2}\, t + \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}\right)$$

$$= 1 - \sin\left(\omega_n t + \pi/2\right)$$

$$= 1 - \sin\left(\pi/2 + \omega_n t\right)$$

$$= 1 - \cos \omega_n t$$

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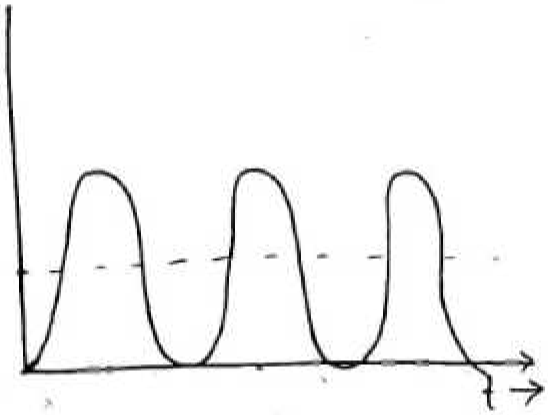

* When $\xi = 0$ $c(t) = 1 - \cos\omega_n t$, from this expression we will get a susten or undamped oscillation.

When $\xi = 1$

$$c(t) = 1 - \frac{e^{\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2}\, t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi^2}\right)$$

$$c(t) = 1 - \frac{e^{\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\omega_n\sqrt{1-\xi^2}\, t \cos\phi + \cos\omega_n\sqrt{1-\xi^2}\, t \sin\phi\right)$$

$$= 1 - \frac{e^{\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\omega_n\sqrt{1-\xi^2}\, t\, \xi + \cos\omega_n\sqrt{1-\xi^2}\, t \cdot \sqrt{1-\xi^2}\right)$$

$$= 1 - \frac{e^{\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\lim_{\xi\to 1}\sin\omega_n\sqrt{1-\xi^2}\, t\, \xi + \lim_{\xi\to 1}\cos\omega_n\sqrt{1-\xi^2}\, t \cdot \sqrt{1-\xi^2}\right)$$

$$= 1 - \frac{e^{\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\omega_n\sqrt{1-\xi^2}\, t \cdot 1 + 1 \cdot \sqrt{1-\xi^2}\right)$$

$$= 1 + \frac{e^{\xi\omega_n t}}{\sqrt{1-\xi^2}}$$

$$= 1 + e^{\xi\omega_n t}$$

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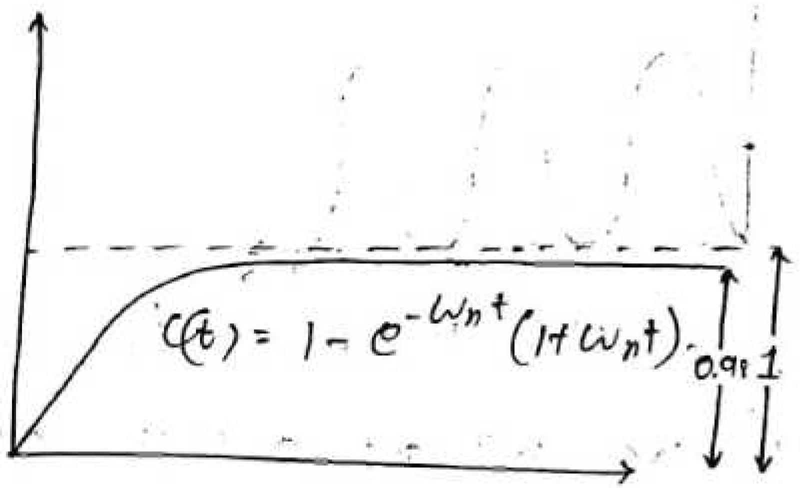
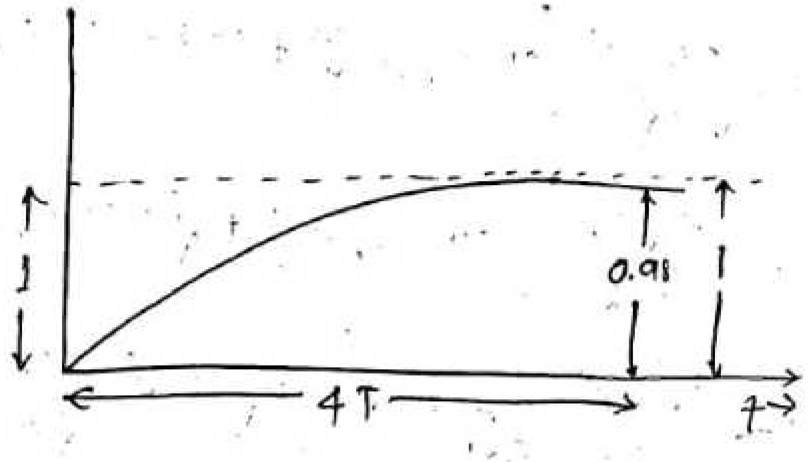
$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$



When $\xi > 1$

When $\xi > 1$ the system becomes over damped.



Critical Damping

From the expression $c(t)$ it is found that $\xi\omega_n$ is responsible for offering the damping in the system.

When $\xi = 0$ oscillations are sustained.
(undamped oscillation)

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For $\zeta < 1$: the oscillation is decay exponentially with time constant $T = \frac{1}{\zeta \omega_n}$.

For $\zeta > 1$ the response doesn't exhibit oscillation, and the response is over damped.

When $\zeta = 1$ the actual damping is ω_n.

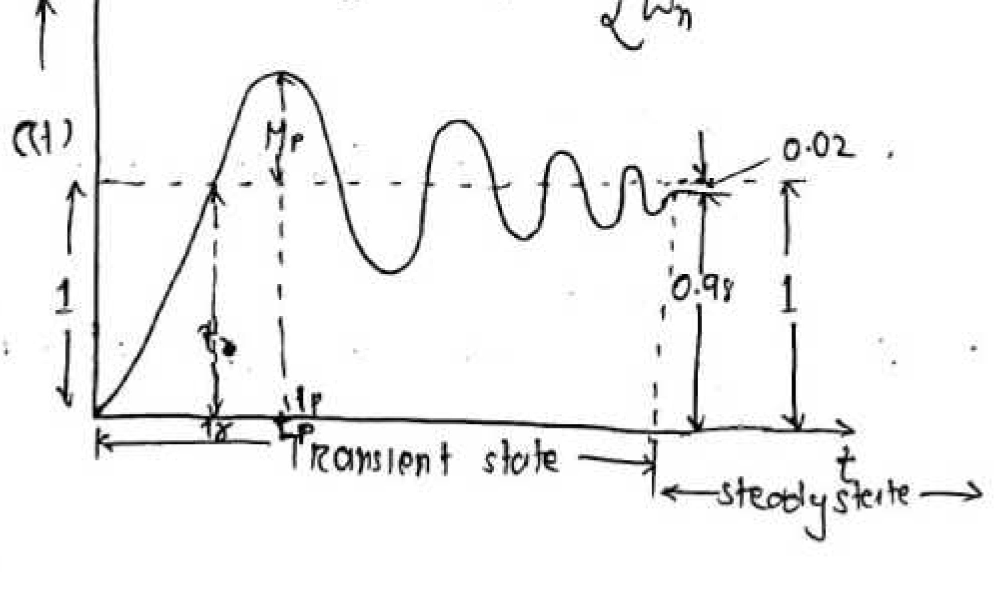
The actual damping when $\zeta = 1$ is called critical damping.

$$\frac{\text{Actual damping}}{\text{critical damping}} = \frac{\zeta \omega_n}{\omega_n} = \zeta = \text{damping ratio}.$$

$\zeta \omega_n \Rightarrow$ damping factor, damping co-efficient or actual damping.

Transient response specification of 2nd order control System

$T = \frac{1}{\zeta \omega_n}$



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* The time response of an under damped control system ~~ex~~ shows oscillation, prior to reach the steady state, with decreasing amplitude.
* To draw a clear idea about the under damped system we should go through some ~~trein~~ transient specification. i.e the (i) rise time, ~~maxir~~ (t_r)
(ii) maximum overshoot $(M_p$
(iii) peak time (t_p)

1) The rise time (t_r)

* The time taken by the response from 0% to reach the 100%. for under damped system is called rise time.
* 10% to 90%. for over damped system

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2}\, t + \phi\right)$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

* When 'ξ' $t = t_r$ $C(t) = 1$

$$1 = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2}\, t + \phi\right)$$

$$\Rightarrow \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2}\, t_r + \phi\right) = 0$$

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But

$$\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}$$ is finite. then

$$\sin(\omega_d t_r + \phi) = 0$$

$$\sin(\omega_d t_r + \phi) = 0$$

$$\omega_d t_r + \phi = \pi$$

$$\boxed{t_r = \frac{\pi - \phi}{\omega_d}}$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

ii) Maximum overshoot (M_P)

The response prior ~~to~~ to reach the steady state oscillate with in (up, down) of the reference input or desired output

∴ The ~~ma~~ maximum +ve deviation from of the actual output w.r.t ~~den~~ desired output is called maximum overshoot.

$$M_P = C(t)_{max} - 1$$

$$\% M_P = C(t)_{max} - 1 \times 100$$

iii) Peak time (t_P)

The time taken by the response or the actual output to reach the maximum overshoot is called peak time.

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When $t = t_p$ $\quad c(t) = c(t)_{max}$.

To get $c(t)_{max}$ the first derivative of $c(t)$ w.r.t t becomes zero.

i.e $\left.\frac{d\,c(t)}{d(t)}\right|_{t=t_p} = 0$.

$$c(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\omega_d t + \phi\right)$$

$$\frac{d\,c(t)}{dt} = 0.$$

$$\Rightarrow \left[\frac{-\xi\omega_n}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t} \sin(\omega_d t + \phi) + \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_d t + \phi).\omega_d\right] = 0$$

$$\Rightarrow \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\xi\omega_n \sin(\omega_d t + \phi) - \omega_d \cos(\omega_d t + \phi)\right] = 0$$

$\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$ is finite.

$$\zeta\omega_n \sin(\omega_d t_p + \phi) = \cos(\omega_d t_r + \phi)\,\omega_d$$

$$\Rightarrow \frac{\sin(\omega_d t_p + \phi)}{\cos(\omega_d t_p + \phi} = \frac{\omega_d}{\zeta\omega_n}$$

$$= \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\,\omega_n}$$

$$\Rightarrow \tan(\omega_d t_p + \phi) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\Rightarrow \tan(\omega_d t_p + \phi) = \tan\phi \quad \left(\because \phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right.$$

$$\Rightarrow \frac{\tan\omega_d t_p + \tan\phi}{1 - \tan\omega_d t_p \cdot \tan\phi} = \tan\phi$$

$$\Rightarrow \tan\omega_d t_p + \tan\phi = \tan\phi - \tan\omega_d t_r \cdot \tan^2\phi$$

$$\Rightarrow \tan\omega_d t_p + \tan\omega_d t_r \cdot \tan^2\phi = \tan\phi \cdot \tan\phi$$

$$\Rightarrow \tan\omega_d t_p\,(1 + \tan^2\phi) = 0$$

But $1 + \tan^2\phi \neq 0$

$$\tan\omega_d t_p = n\pi$$

For first over shoot $n = 1$

$$\omega_d t_p = \pi$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

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Maximum overshut (M_P)

$$c(t)_{max} = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t_p + \phi\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \phi\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\pi + \phi)$$

$$= 1 - \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} (-\sin\phi)$$

$$= 1 + \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{1-\zeta^2} \sin\phi$$

$$= 1 + \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \left(\sqrt{1-\zeta^2}\right)$$

$$c(t)_{max} = 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\boxed{c(t)_{max} = 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}$$

$$M_p = c(t)_{max} - 1$$

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$$= 1 + \frac{e^{-\xi\pi}}{\sqrt{1-\xi^2}} - 1$$

$$= \frac{e^{-\xi\pi}}{\sqrt{1-\xi^2}}$$

$$\%\, M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100$$

Formula

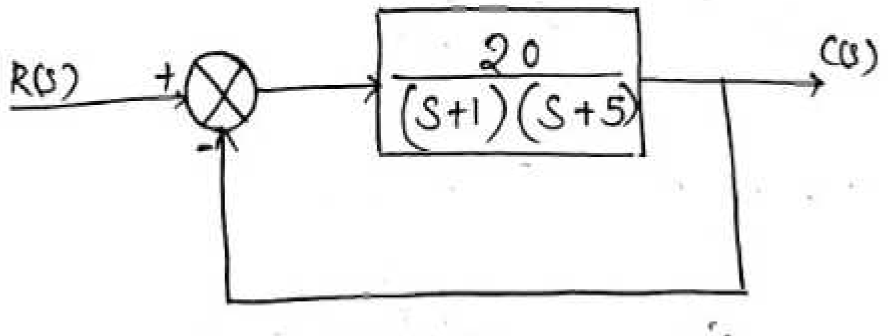
$$t_r = \frac{\pi - \phi}{\omega_d} \qquad \phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

$$t_p = \frac{\pi}{\omega_d} \qquad \omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\%\, M_p = e^{\frac{-\xi\pi}{1-\xi^2}} \cancel{\times 100} \times 100$$

$$C(t)_{max} = 1 + e^{\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

Prob



From the block diagram of a unit feedback control system determine the characteristics equation, ω_n, ξ, ω_d, t_r, t_p, M_p, the time at which the first under shut occure, time period of oscillation, time taken to reach the steady state.

Characteristics eq^n

$$(S+1)(S+5)=0$$

$$\Rightarrow S^2+6S+$$

$$\Rightarrow S^2+5S+S+5=0$$

$$\Rightarrow S^2+6S+5=0$$

$$G(s)=\frac{20}{(s+1)(s+5)}=\frac{20}{S^2+6s+5}=\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$H(s)=1$$

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s).H(s)}$$

$$=\frac{\frac{20}{s^2+6s+5}}{1+\frac{20}{s^2+6s+5}}$$

$$\frac{C(s)}{R(s)}=\frac{20}{s^2+6s+25}$$

Ch: eq^n: $s^2+6s+25$

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\omega_n^2=25$$

$$\Rightarrow \omega_n=5$$

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$2\zeta\omega_n = 6$

$\Rightarrow 2 \times 5\, \zeta = 6$

$\Rightarrow \zeta = \frac{6}{10} = 0.6$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

$= 5\sqrt{1-(0.6)^2}$

$= 4$

$\zeta\omega_n = 0.6 \times 5 = 3$

$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{0.8}{0.6}\right)$

$= \tan^{-1}\left(\frac{4}{3}\right)$

$= \tan^{-1}(1.33)$

$= 53.13^\circ$

$= 0.92$

$\pi^c = 180^\circ$

$180^\circ = \pi^c$

$1^\circ = \frac{\pi}{180}$

$\theta = \frac{\pi}{180} \times \theta$

$t_r = \frac{\pi - \phi}{\omega_d}$

$= \frac{3.142 - 0.92}{4}$

$= 0.55$ sec.

$t_p = \frac{\pi}{\omega_d} = \frac{3.142}{4} = 0.78$ sec

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$$\% \ M_p = e^{\frac{-0.6 \times 3.142}{0.8}} = 0.094 = 9.4\%$$

Time to reach the first undershoot is

$$t = \frac{2\pi}{\omega_d} = 2 \times 0.785 = 1.57 \text{ sec.}$$

$$T = \frac{1}{\xi\omega_n} = \frac{1}{0.6 \times 5} = \frac{1}{3} = 0.33 \text{ sec.}$$

Time taken to reach the steady state

$$= 4T = \frac{4}{\xi\omega_n} = 4 \times \frac{1}{3} = \frac{4}{3} = 1.33 \text{ sec.}$$

Prob

$$G(s) = \frac{25}{s(s+10)}$$

$$H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s).H(s)}$$

$$= \frac{\frac{25}{s^2+10s}}{1 + \frac{25}{s^2+10s}}$$

$$= \frac{\frac{25}{s^2+10s}}{\frac{s^2+10s+25}{s^2+10s}}$$

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$$= \frac{25}{s^2+10s+25} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Characteristics eqn

$$= s^2+10s+25 = 0 \quad --(1)$$

$$\omega_n^2 = 25$$

$$\Rightarrow \omega_n = 5 \text{ rad/se.}$$

$$2\zeta\omega_n = 10$$

$$\Rightarrow 2\times5\zeta = 10$$

$$\Rightarrow \zeta = \frac{10}{10} = 1$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$= \cancel{5\times}\,0$$

$$t_p = \frac{\pi}{\omega_d} = \frac{3.142}{0} = \infty$$

$$M_p = e^{-\frac{\zeta\pi}{1-\zeta^2}}$$

$$= \cancel{\infty}\; e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

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Steady State error

Steady State error is the difference between the desired output or refference input to the actual output at steady state.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} \left[r(t) - c(t) \right]$$

$$e_{ss} = \lim_{s \to 0} s \cdot E(s)$$

* The steady state error gives the index of accuracy of a control system
* The steady state error should be minimum the magnitude of steady state error in a close loop control system depends upon the open loop transfer function i.e $G(s).H(s)$

Classification of Open loop tranfer function

$$G(s).H(s) = \frac{K(1+sT_a)(1+sT_b)\ldots}{s^N (1+sT_1)(1+sT_2)\ldots}$$

K = Forward path gain.

$= -\frac{1}{T_a}, -\frac{1}{T_b} \ldots$ are zeros.

$-\frac{1}{T_1}, -\frac{1}{T_2} \ldots$ are poles,

A

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Here N is the number of poles at origin.

* The N will define the types of open loop transfer function. i.e the close loop control system.

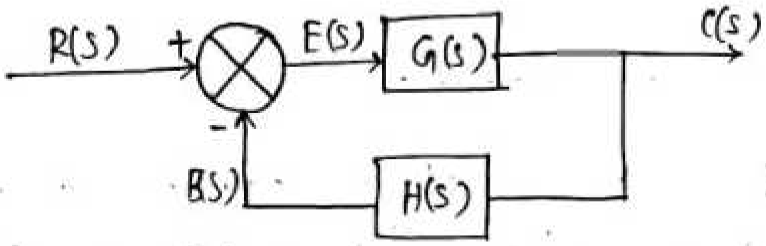
* If $N=0$, there is no poles at origin.

$N = 0$, the type '0' system.

$N = 1$, the type '1' system

$N = 2$, the type 2 system.

$N = N$, the type N system.



$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S).H(S)}$$

$$\frac{E(S)\,G(S)}{R(S)} = \frac{G(S)}{1+G(S)\,H(S)}$$

$$\Rightarrow \frac{E(S)}{R(S)} = \frac{1}{1+G(S)\,H(S)}$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim \left[r(t) - c(t)\right]$$

$$e_{ss} = \lim_{S\to 0} S.E(S)$$

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$$= \lim_{s \to 0} s \frac{R(s)}{1+G(s).H(s)}$$

From this expression it is found that the steady state error can be obtained from reference input & openloop transfer function.

The actual output of a control system may be in physical form is called as position or displacement.

The first derivative of the displacement is velocity and the 2nd derivative of displacement is called acceleration.

Static Error co-efficient :

As the steady state error is obtained in steady state period the error is called static error.

Steady state error is called static error and it is associated with static error coefficient.

Static Positional Error co-efficient (K_p)

The static positional Error co-efficient k_p is associated with unit step input applied to a closed loop control system.

$$e_{ss} = \lim_{s \to 0} s.E(s) = \lim_{s \to 0} s\, R(s) \cdot \frac{1}{1+G(s).H(s)}$$

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Unit step input $= r(t) = 1$

$r(s) = \frac{1}{s}$

$$e_{ss} = \lim_{s \to 0} s \times \frac{1}{s} \frac{1}{1+G(s).H(s)}$$

$$= \frac{1}{1 + \lim_{s \to 0} G(s)\,H(s)}$$

$$e_{ss} = \frac{1}{1+k_p}$$

$$\boxed{k_p = \lim_{s \to 0} G(s).H(s)}$$

Static Velocity error coefficient (K_v)

The static velocity error coefficient is associated with unit ramp input applied to a close loop control system.

Unit ramp input $r(t) = t$

$R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s \to 0} s \times \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \to 0} s \times \frac{1}{s^2} \frac{1}{1+G(s).H(s)}$$

$$= \lim_{s \to 0} \frac{1}{s + s\,G(s).H(s)}$$

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$$= \frac{1}{\lim\limits_{s\to 0} s + \lim\limits_{s\to 0} s\, G(s).H(s)}$$

$$= \frac{1}{\lim\limits_{s\to 0} sG(s)H(s)}$$

$$C_{ss} = \frac{1}{K_V}$$

$$\boxed{K_V = \lim_{s\to 0} s\, G(s).H(s)}$$

Static Acceleration error co-efficient (K_a)

Static accelaration error co-efficient (K_a) is assciated with unit parabolic input applied with close loop control system:

Unit Parabolic input $r(t) = \frac{t^2}{2}$

$$R(s) = \frac{1}{s^3}$$

$$C_{ss} = \lim_{s\to 0} s \times \frac{1}{s^3} \frac{1}{1+G(s)H(s)}$$

$$= \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s).H(s)}$$

$$= \frac{1}{\lim\limits_{s\to 0} s^2 + \lim\limits_{s\to 0} s^2 G(s).H(s)}$$

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$$= \frac{1}{\lim_{s \to 0} s^2 G(s).H(s)}$$

$$e_{ss} = \frac{1}{k_a}$$

$$\boxed{k_a = \lim_{s \to 0} s^2 G(s).H(s)}$$

Type '0' system :

No poles at origin.

i.e $N = 0$

$$G(s).H(s) = \frac{k(1+s_aT)(1+s_bT)\dots}{s^N(1+s_1T)(1+s_2T)\dots}$$

for

D When unit step input is given :-

$$K_p = \lim_{s \to 0} G(s).H(s)$$

$$= \lim_{s \to 0} \frac{K(1+s_aT)(1+s_bT)\dots}{(1+s_1T)(1+s_2T)\dots}$$

$$\boxed{K_p = K}$$

static or positional error coefficient for a type '0' system is equal to forward path gain.

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Steady state error:-

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K}$$

$$\boxed{e_{ss} = \frac{1}{1+K}}$$

* The steady state error is finite when unit step input is given to the type '0' system.. so this is acceptable

2) When unit ramp input is given:

Static velocity error coefficient

$$K_v = \lim_{S\to 0} SG(S).H(S)$$

$$= 0$$

Steady state error :-

$$e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

$$\boxed{e_{ss} = \infty}$$

* The static velocity error coefficient is zero and steady state error is infinite. the system is not acceptable for a ramp input.

3) When unit parabolic input is given.

Static acceleration error coefficient

$$K_a = \lim_{S\to 0} S^2 G(S).H(S)$$

$$= 0$$

Steady state error

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

$$\boxed{e_{ss} = \infty}$$

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* The static accelaration error coefficient is 0. and steady state error is ∞. the system is not acceptable for a unit parabolic input.

Type '1' system

1 pole at origin

i.e $N = 1$

$$G(s).H(s) = \frac{K(1+S_aT)(1+S_bT)\ldots}{S(1+S_1T)(1+S_2T)\ldots}$$

I) When unit step input is given

static positional error coefficient

$$K_p = \lim_{s\to 0} G(s).H(s)$$

$$= \lim_{s\to 0} \frac{K(1+S_aT)(1+S_bT)\ldots}{S(1+S_1T)(1+S_2T)\ldots}$$

$$= \infty$$

Steady state error

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = \frac{1}{\infty} = 0$$

$$\boxed{e_{ss} = 0}$$

* Static positional error coefficient is infinite and the steady state error is '0', so the system is acceptable for a unit step input is given to type 1 system.

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2) When unit Ramp input is given

Static velocity error co-efficient (K_v)

$$K_v = \lim_{s \to 0} s\, G(s) \cdot H(s)$$

$$= \lim_{s \to 0} s \frac{K(1+s_aT)(1+s_bT)\cdots}{s(1+s_1T)(1+s_2T)\cdots}$$

$$K_v = K$$

Steady state error (e_{ss})

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}$$

$$\boxed{e_{ss} = \frac{1}{K}}$$

State velocity erro coefficient is k and the steady state error is finite so the unit ramp system is acceptable for unit ramp input to the type '1' system,

3) When unit Parabolic is given:

Static accelaration error coefficient

$$K_a = \lim_{s \to 0} s^2\, G(s)\, H(s)$$

$$= \lim_{s \to 0} = 0$$

Steady state error (e_{ss})

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

$$\boxed{e_{ss} = \infty}$$

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static accelaration error coefficient is zero and the steady state error is ∞ so the system is not acceptable for unit ramp parabolic input to the type '2' system.

Type - 2 system

Two pole at origin

$N = 2$

$$G(s) \cdot H(s) = \frac{K(1+S_aT)(1+S_bT)\cdots}{S^2(1+S_1T)(1+S_2T)\cdots}$$

When unit step input is given

Static positional error co-efficient

$$K_p = \lim_{s \to 0} G(s) \cdot H(s)$$

$$= \lim_{s \to 0} \frac{K(1+S_aT)(1+S_bT)\cdots}{S^2(1+S_1T)(1+S_2T)\cdots}$$

$K_p = \infty$

Steady state error

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

$e_{ss} = 0$

* static positional error co-efficien is ∞ and the steady state error is 0, so the system is acceptable for unit step input to the type '2' system.

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2) When unit ramp input is given

Static velocity error co-efficient

$$K_v = \lim_{s \to 0} s\, G(s) \cdot H(s)$$

$$= \lim_{s \to 0} s \frac{K(1+saT)(1+sbT)}{s^2(1+s_1T)(1+s_2T)}$$

$$K_v = \infty$$

✓

Steady state error

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

* Static velocity error co-efficient is ∞ and the ~~static~~ steady state error is 0, so the system is ~~acceptab~~ acceptable for unit ramp input to the type '2' system

3) When unit parabolic input is given

Static acceleration error co-efficient

$$K_a = \lim_{s \to 0} s^2\, G(s) \cdot H(s)$$

$$K_a = \lim_{s \to 0} \cancel{s^2} \frac{K(1+saT)(1+sbT) \ldots}{\cancel{s^2}(1+s_1T)(1+s_2T) \ldots}$$

$$\boxed{K_a = K}$$

Steady state error

$$e_{ss} = \frac{1}{K_a} = \frac{1}{K}$$

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~~∴~~ static accelaration error co-efficient is k and steady state error is finite so the system is ~~not~~ acceptable for unit parabolic input to the type '2' system.

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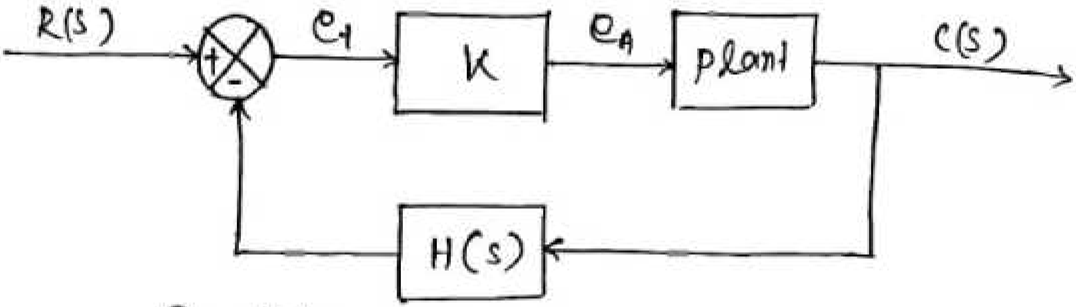
Control Action

$$e_t = r(t) - c(t)$$

Proportional Controller



$$e_A = ke_t$$

* In proportional control the actuating signal(e_A) for the control action is proportional to error signal (e_t)

i.e $e_A \propto e_t$

$$e_A = ke_t$$

K is the constant forward path gain.

* The error signal is the difference between the reference input signal and the feedback output signal.
* It is always desirable control system must be under damped with oscillating output and decreasing amplitude. and the system is fast
* For sluggish over damped system to made faster by increasing its forward path gain K. As a result the steady state error decrease

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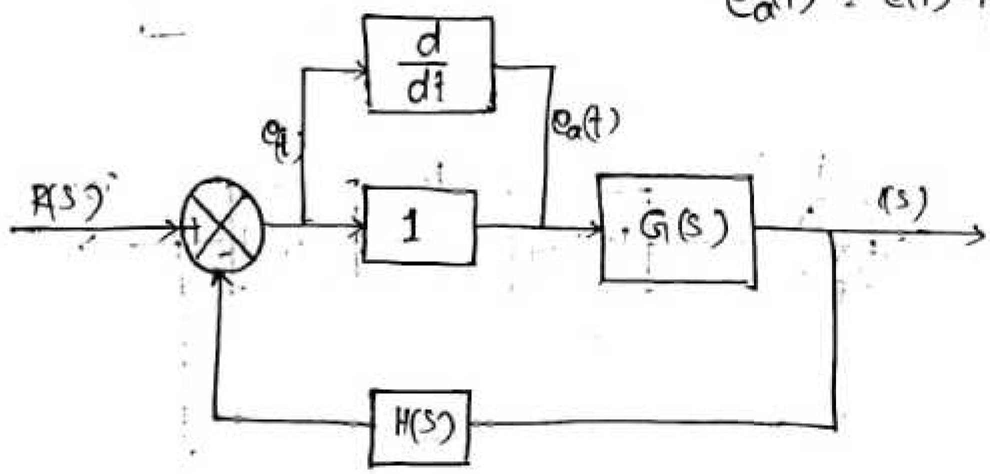
* At the same time the maximum overshoot increases.

* For satisfactory performance of a control system a proper adjustment has to be made between the peak overshoot and the steady state error.

Proportional Derivative Controller

$$e_a(t) = e(t) + \frac{d}{dt} e(t)$$



In PD controller the actuating signal is proportional to error signal plus proportional to derivative of error signal

$$e_a(t) = e(t) + \frac{d}{dt} e(t)$$

Taking laplace transform on both side

$$E_a(S) = E(S) + S T_d E(S)$$

$$E_a(S) = E(S)(1 + S T_d)$$

$e_a(t) \propto e(t)$

$e_a(t) = k e(t)$

$e_a(t) \propto \frac{d}{dt} e(t)$

$e_a(t) = k \frac{d}{dt} e(t)$

$\propto \frac{d}{dt} e(t)$

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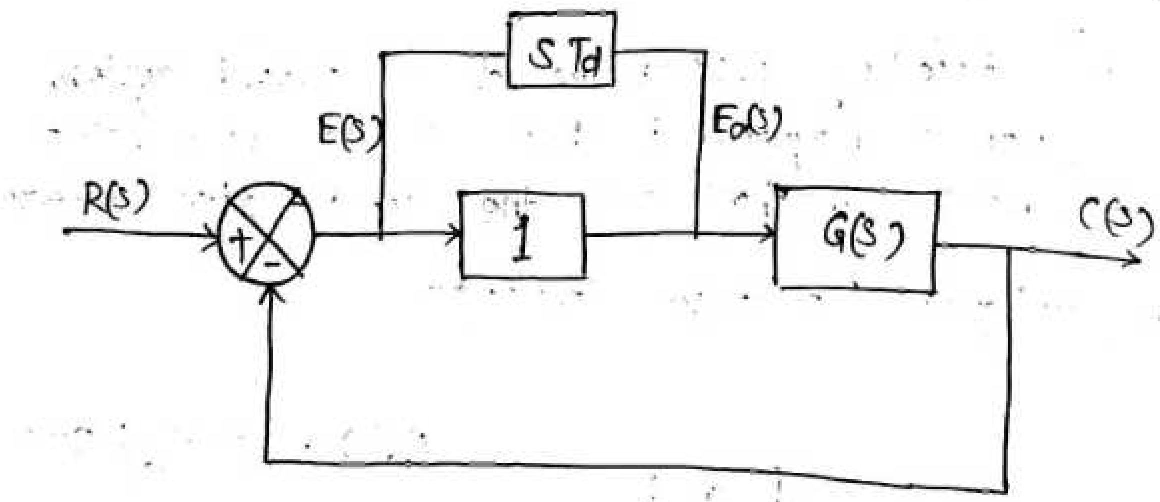

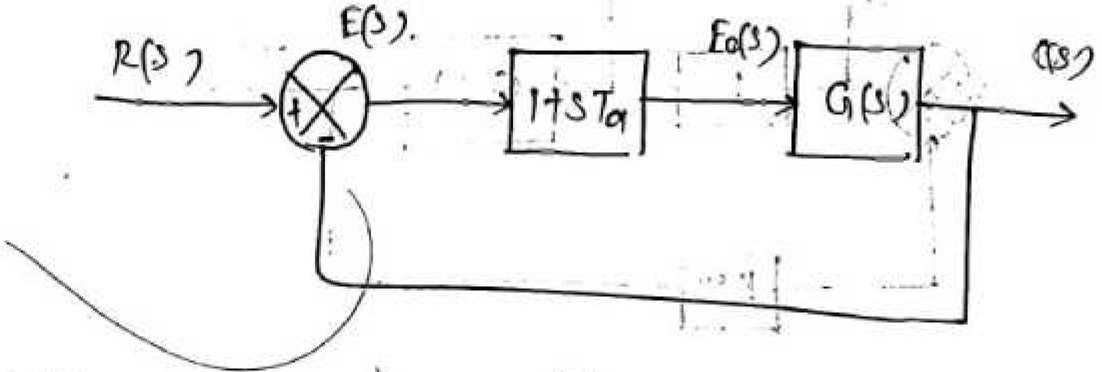

$$G(s) = \frac{\omega_n}{s^2+2\zeta\omega_n s}$$

$$G'(s) = (1+sT_d)\frac{\omega_n}{s^2+2\zeta\omega_n s}$$

$$\frac{C(s)}{R(s)} = \frac{G'(s)}{1+G'(s)\,H(s)}$$

$$= \frac{(1+sT_d)\cdot\frac{\omega_n}{s^2+2\zeta\omega_n s}}{1+(1+sT_d)\left(\frac{\omega_n}{s^2+2\zeta\omega_n s}\right)}$$

$$= \frac{(1+sT_d)\,G(s)}{1+(1+sT_d)\,G(s)}$$

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$$= \frac{(1+sT_d)\omega_n^2}{s^2+2\zeta\omega_n s+(1+sT_d)\omega_n^2}$$

$$= \frac{(1+sT_d)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2+\omega_n^2 sT_d}$$

$$\frac{C(s)}{R(s)} = \frac{(1+sT_d)\omega_n^2}{s^2+s(2\zeta\omega_n+T_d\omega_n^2)s+\omega_n^2} \quad \text{--- (i)}$$

$$= \frac{(1+sT_d)\omega_n^2}{s^2+(2\zeta'\omega_n)s+\omega_n^2} \quad \text{--- (ii)}$$

Comparing eq^n (i) & (ii)

$$2\zeta'\omega_n s = (2\zeta+T_d\omega_n)s\omega_n$$

$$2\zeta' = 2\zeta+T_d\omega_n$$

$$\boxed{\Rightarrow \zeta' = \zeta+\frac{\omega_n T_d}{2}}$$

$$\therefore \zeta' > \zeta$$

* In PD controller $\zeta' = \zeta+\frac{\omega_n T_d}{2}$ i.e $\zeta' > \zeta$ as damping ratio increases the maximum overshoot is also decreases which is desirable.

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$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s).H(s)}$$

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s} \qquad H(s) = 1$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{\omega_n^2}{s^2+2\zeta\omega_n s}}$$

$$= \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$E(s) = R(s)\left(\frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

For ~~R(t)~~ $r(t) = t$ $\quad R(s) = \frac{1}{s^2}$ for ramp i/p

$$e_{ss} = \lim_{s\to 0} s\,E(s)$$

$$= \lim_{s\to 0} s^2 \cancel{x} \frac{1}{s^2} \frac{s^2\,2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \lim_{s\to 0} \cancel{s^2}\times\frac{1}{\cancel{s^2}} \frac{2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n 2}$$

$$= \lim_{s\to 0} \frac{2\zeta\omega_n}{\omega_n^2}$$

$$e_{ss} = \frac{2\zeta}{\omega_n}$$

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$$E(s) = \frac{1}{1+G'(s).H(s)}$$

$$G' = (1+sT_d)\,G(s) \quad H(s) = 1$$

$$E(s) = R(s)\frac{1}{1+G'(s)\;H(s)}$$

$$e_{ss} = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\frac{R(s)}{1+G'(s).H(s)}$$

$$e_{ss} = \lim_{s\to 0} s\frac{1}{s^2}\frac{1}{1+(1+sT_d)\frac{\omega_n^2}{s^2+2\xi\omega_n s}\times 1}$$

$$= \lim_{s\to 0} s\frac{1}{s^2}\frac{(s^2+2\xi\omega_n s)}{s^2+2\xi\omega_n s+sT_d\omega_n^2+\omega_n^2}$$

$$= \lim_{s\to 0} s^2\times\frac{1}{s^2}\frac{s(s+2\xi\omega_n)}{s^2+2\xi\omega_n s+sT_d\omega_n^2+\omega_n^2}$$

$$= \frac{2\xi\omega_n}{\omega_n^2} = \frac{2\xi}{\omega_n}$$

$$e_{ss} = \frac{2\xi}{\omega_n}$$

The steady state error of a second order control system with unit ramp input without P.D control is $\left(e_{ss} = \frac{2\xi}{\omega_n}\right)$

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* The steady state error of a second order order control system with proportional derivative control action when unit ramp input is applied i.e $e_{ss} = \frac{2\xi}{\omega_n}$ i.e there is no change in steady state error i.e the effect of PD controller on steady state error is zero.

Proportional Intrignal Controller

In intigral control action the actuating signal proportional to the sum of error signal and integral of the error signal.

$$e_a(t) \propto e(t) + k_i \int e(t) \cdot dt$$

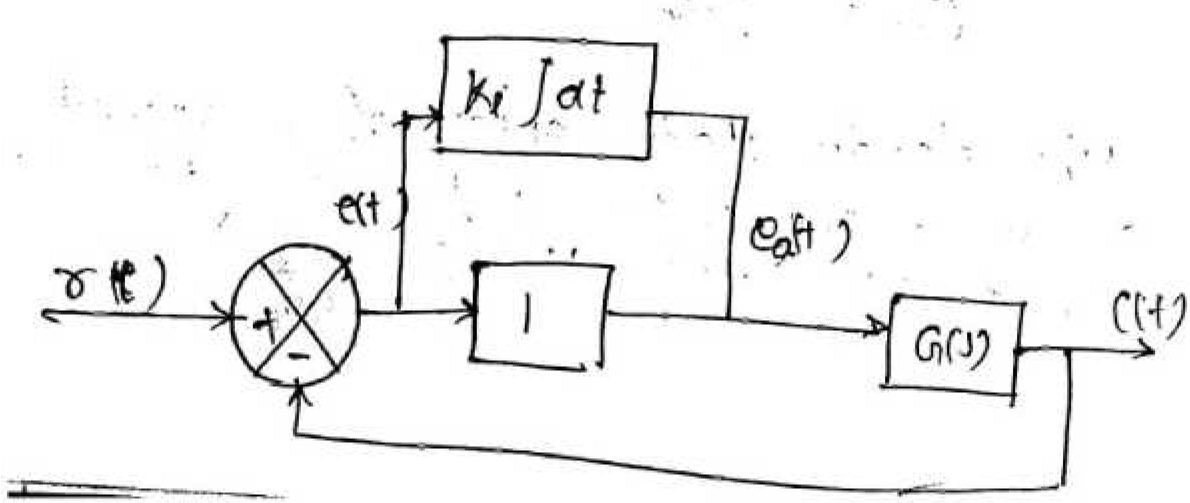
$$e_a(t) = k \left[e(t) + k_i \int e(t) \cdot dt \right]$$

Where $k = 1$

Taking Laplace transform.

$$E_a(s) = E(s) + \frac{k_i}{s} E(s)$$

$$= E(s)\left(1 + \frac{k_i}{s}\right)$$



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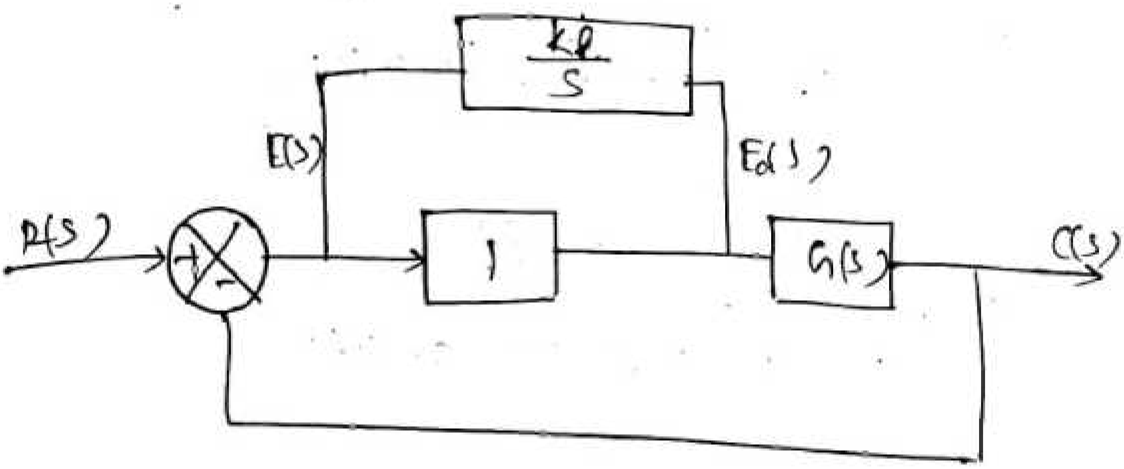

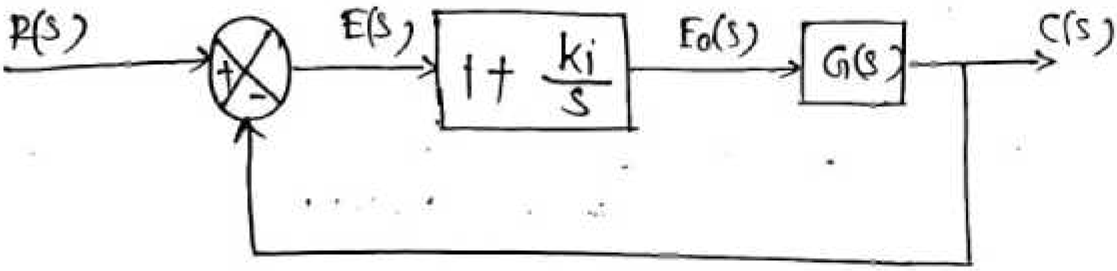

$$G'(s) = \left(1+\frac{ki}{s}\right)G(s)$$
$$H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G'(s)}{1+G'(s)H(s)}$$

$$= \frac{\left(1+\frac{ki}{s}\right)(G(s))}{1+\left(1+\frac{ki}{s}\right)G(s)}$$

$$= \frac{\left(\frac{s+ki}{s}\right) G(s)\ \frac{\omega_n^2}{s^2+2\xi\omega_n s}}{1+\frac{s+ki}{s}\ \frac{\omega_n^2}{s^2+2\xi\omega_n s}}$$

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$$= \frac{\omega_n^2(s+k_i)}{s(s^2+2\zeta\omega_n s)+(s+k_i)\omega_n^2}$$

$$= \frac{\omega_n^2(s+k_i)}{s^3+2\zeta\omega_n s^2+\omega_n s+k_i\omega_n^2}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G'(s)H(s)}$$

$$= \frac{1}{1+\left(1+\frac{k_i}{s}\right)G(s)\times 1}$$

$$= \frac{1}{1+\frac{s+k_i}{s}\,\frac{\omega_n^2}{s^2+2\zeta\omega_n s}\times 1}$$

$$= \frac{s^3+2\zeta\omega_n s^2}{s^3+2\zeta\omega_n s^2+(s+k_i)\omega_n^2}$$

$$e_{ss} = \lim_{s\to 0} s\,E(s)$$

$$= \lim_{s\to 0} s\,R(s)\,\frac{s^3+2\zeta\omega_n s^2}{s^3+2\zeta\omega_n s^2+(s+k_i)\omega_n^2}$$

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For unit ramp input. $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s \to 0} s \times \frac{1}{s^2} \frac{s^3 + 2\xi\omega_n s^2}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + k_i\omega_n^2}$$

$$= \lim_{s \to 0} \cancel{s^2} \times \frac{1}{\cancel{s^2}} \frac{s^2 + 2\xi\omega_n s}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + k_i\omega_n^2}$$

$$= \frac{0}{k_i\omega_n^2} = 0$$

* The steady state error of a 2nd order control system with intignal controller when unit ramp input is applied is zero (0)

* So the intignal control action reduces the steady state error.

Proportional Intignal Derivative Controller (PID)

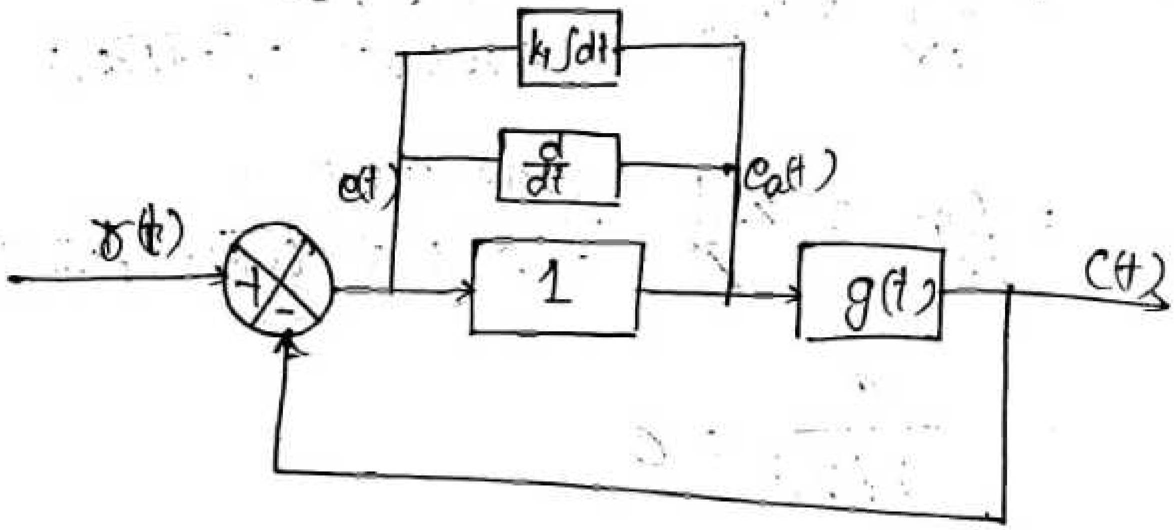
* For PID control the actuating signal consist of proportional error signal plus derivative of error signal plus intignal of error signal.

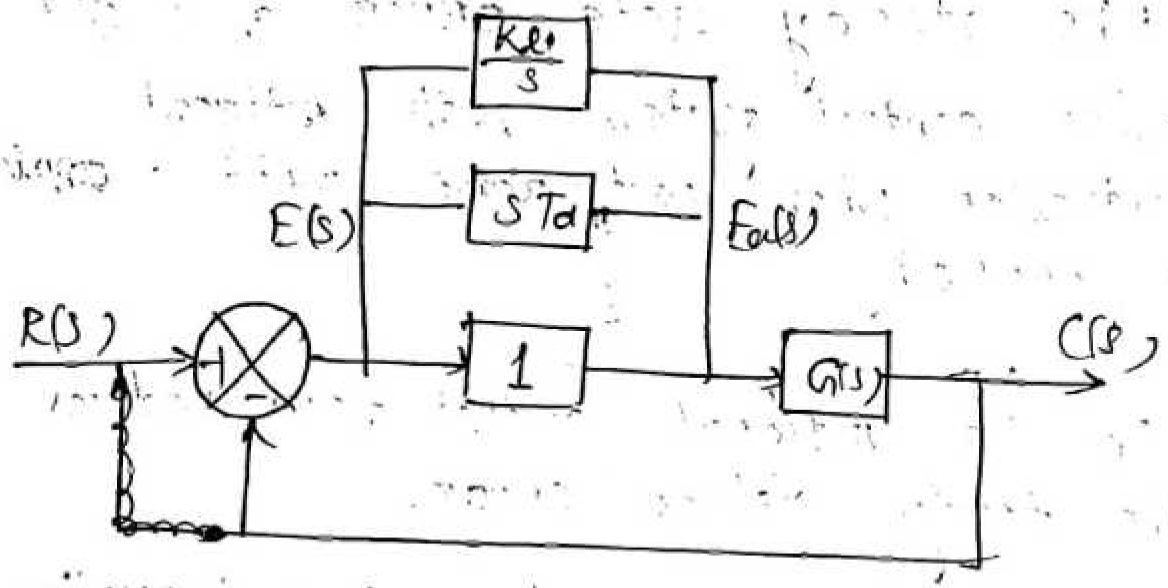
$$e_a(t) = e(t) + T_d \frac{d}{dt} e(t) + k_i \int e(t)\,dt$$

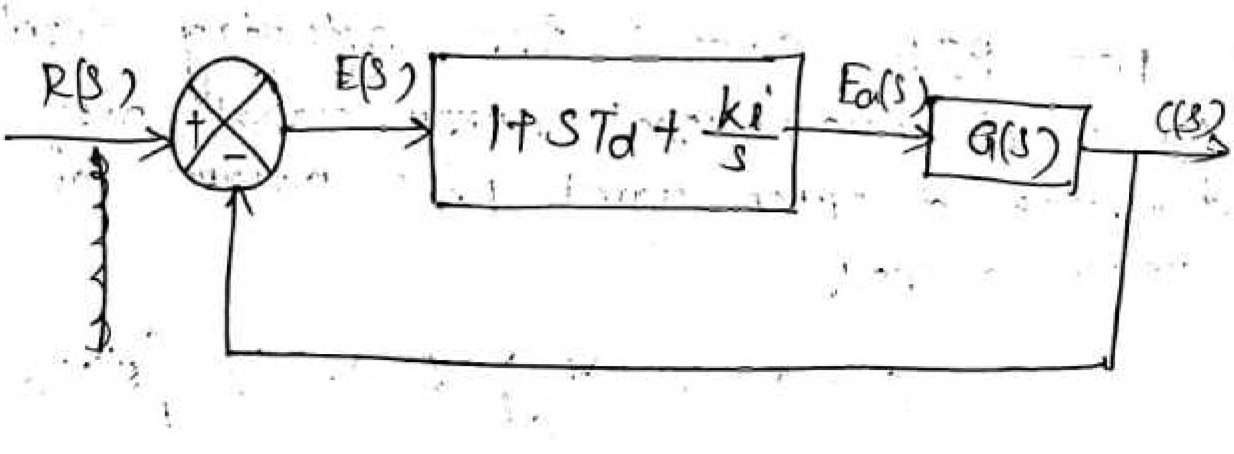
Taking laplace transfer.

$$E_a(S) = E(S) + S T_d E(S) + \frac{k_i}{s} \frac{E(S)}{s}$$

$$= \left(1 + sT_d + \frac{k_i}{s}\right) E(s)$$







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The PID control does the combined effect of proportional derivative & Intignal control.

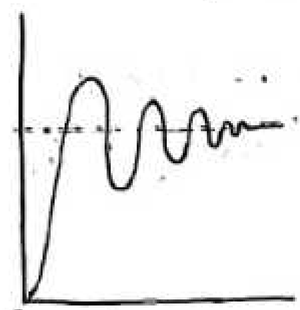
Stability :-

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

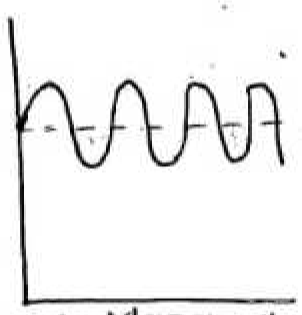
$$G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)}$$

$$H(s) = 1 .$$

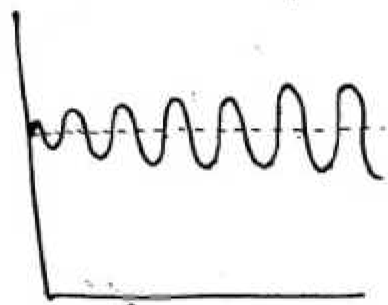
* The stability analysis can be assist qualitatively by knowing system characteristics equation or transfer function.
* After application of an input the output of the control system is oscillatory and damped out with respect to time the system is called stable system.
* If the amplitute of oscillation is sustain is called marginally stable.
* If the amplitute of osillation is increases the system is called unstable.
*



(Stable System)



(Marginally stable System)



(Unstable System)

Absolute Stability

If the system is stable is always irrespective of the different condition is called absolute stability.

The absolute stability can be determine from the qualitative analysis. i.e location of the roots of the characteristics eqn in 'S' plane.

Relative Stability

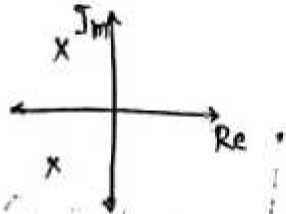
Relative stability is used in reation to compaurative analysis of stability.

- The relative stability can be determine from the maximum overshoot, Gain margine & phase margine, damping ratio.

Absolute Stability :-

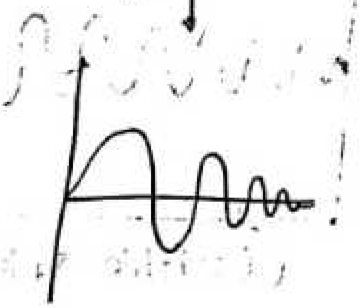
The system is called absolute stable if the real parts of roots of the characteristics eqn are all negative;

If one of the roots having (+ve) real part then the system is unstable.



(System is stable)

Roots of the characteristics eqn having -ve real parts.



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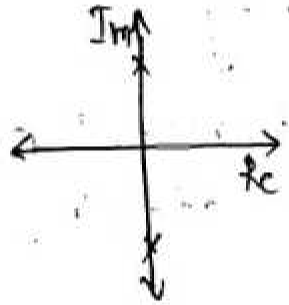
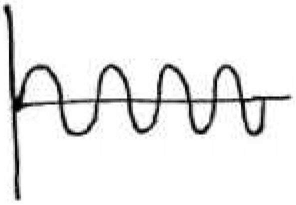
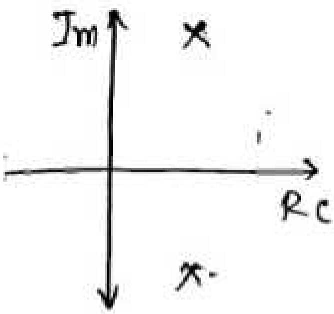
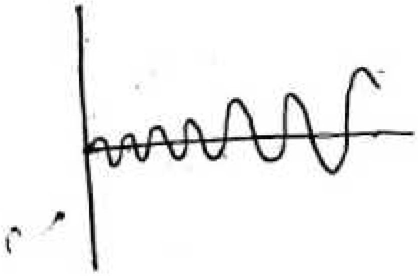


(System is marginally stable)
Roots of the characteristic eqn are zero or roots are lies in imaginary axis and complex conjugate to each other.





(System is unstable)
Roots of the eqn having +ve real parts.



The characteristics eqn is given by

$1 + G(s) \cdot H(s)$.

In polynomial form.

$$a_0 s^n + a_1 s^{n-1} + \dots + a_n = 0$$

Necessary condition for stability

* All the ~~po~~ coefficients of the above polynomial must be positive.

*

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Hurwitz criteria for stability :

If all the Hurwitz determinants are (+ve) then the system is stable.

Characteristics eqn

$$1 + G(s).H(s) = 0$$

In polynomial form.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_n = 0$$

$$\Delta_1 = a_1 > 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0$$

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2n-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2n-2} \\ 0 & a_1 & a_3 & \cdots & a_{2n-3} \\ 0 & 0 & a_0 & \cdots & a_{2n-4} \\ 0 & 0 & 0 & & a_n \end{vmatrix} > 0$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_4 & a_3 \end{vmatrix}$$

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$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & a_0 & a_2 \end{vmatrix}$$

Routh Hurwitz criteria for stability

Charecteristics (eqn)

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0$$

$$\Delta_n = \begin{vmatrix} a_0 & a_2 & a_4 & \cdots & a_{2n+2} \\ a_1 & a_3 & a_5 & \cdots & a_{2n+1} \\ b_1 & b_3 & b_5 & \cdots & b_n \\ c_1 & c_3 & c_5 & \cdots & c_n \\ d_1 & d_3 & d_5 & \cdots & d_n \end{vmatrix}$$

Write down the eh

* The sign of the number present in the first column of the routh Hurwitz determinant same and there is no sign change then the system is stable

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_3 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_5 - a_1 b_5}{b_1}$$

Prob

A close loop control system has the characteristics eqn $s^3 + 4.5s^2 + 3.5s + 1.5 = 0$ investigate the stability using routh Harwitz array.

Soln The characteristics eqn is given by.

$$s^3 + 4.5s^2 + 3.5s + 1.5 = 0$$

~~$s^3 +$ a_0.~~

~~Necessa~~

Checking of necessary condition of stability

1) There is no missing term.
2) All the coefficients of 's' are (+ve)

The system may be stable.

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$$
\begin{array}{c|cc|}
s^3 & 1 & 3.5 \\
s^2 & 4.5 & 1.5 \\
s^1 & \dfrac{1\times1.5-3.5\times4.5}{4.5} = -3.16 & 0 \\
s^0 & 1.5 & 0
\end{array}
$$

* In routh ~~how~~ Harwitz determinant there is 2 times sign change in the element of the 1st column that means 2 roots having (+ve) real parts.

$\cancel{s^3 + 4.5 s^2 + 3.5 s + 1.5 = 0}$

Prob

$s^3 + 4\times10^2 s^2 + \cancel{5+5\times10^4}\ 5\times10^4 s + 2\times10^6 = 0$

Put $s = 10^2 p$

$(10^2 p)^3 + 4\times10^2 (10^2 p)^2 + 5\times10^4 (10^2 p) + 2\times10^6 = 0$

$\Rightarrow 10^6 p^3 + 4\times10^6 \cancel{\times 10^4} \times p^2 + 5\times10^6\, p + 2\times10^6 = 0$

$\Rightarrow 10^6 (p^3 + 4p^2 + 5p + 2) = 0$

$\Rightarrow p^3 + 4p^2 + 5p + 2 = 0$

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p^3	1	5
p^2	4	2
p^1	4.5	0
p^0	2	0

$$b_1 = \frac{4\times5 - 2\times1}{4}$$

$$= \frac{20-2}{4} = \frac{18}{4} = 4.5$$

$$b_3 = 0$$

$$c_1 = \frac{4.5\times2}{4.5} = 2$$

$$c_3 = 0$$

Proplems in Routh Hurwitz array.

1) If one of the element of 1st column of Routh Hurwitz determinant is zero. and atleast one (+ve) number in that row.

$$\underset{a_0}{s^4} + \underset{a_1}{3}s^3 + \underset{a_2}{s^2} + \underset{a_3}{3}s + \underset{a_4}{2} = 0$$

s^4	1	1	2
s^3	3	3	0
s^2	$0(\epsilon)$	2	0
s^1	$\frac{3\epsilon-3\times2}{\epsilon}$	0	0
s^0	2	0	0

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put ϵ in place of zero (0).
where $\epsilon \to 0$

$$C_1 = \frac{3 \times \epsilon - 3 \times 2}{\epsilon}$$

$$= \frac{3\epsilon - 6}{\epsilon}$$

Prob.

2) If all the elements of one of the rows of the Routh Harwitz determinant are zeros

$$s^5 + 6s^4 + 6s^3 + 12s^2 + 5s + 6 = 0$$

s^5	1	6	5
s^4	6	12	6
s^3	4	4	0
s^2	6	6	0
s^1	12	0	0
s^0	6	0	0

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$$b_1 = \frac{6 \times 6 - 12}{6}$$

$$= \frac{36 - 12}{6} = \frac{\cancel{24}^{4}}{\cancel{6}} = 4$$

$$b_3 = \frac{6 \times 5 - 6}{6}$$

$$= \frac{30 - 6}{6} = 4$$

$$c_1 = \frac{12 \times 4 - 24}{6}$$

$$= \frac{48 - 24}{4} = 6$$

$$c_3 = \frac{24 - 6}{4} = 6$$

Take the Row above the row having all elements zero (0)

$$A = 6s^2 + 6$$

$$\frac{dA}{ds} = \frac{d}{ds}\left(6s^2 + 6\right)$$

$$= \frac{d}{ds} 6s^2 + \cancel{\frac{d}{ds} 6}$$

$$= 12s + \cancel{6}$$

put the co-efficient

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Root locus

$$G(s)\ H(s) = -1$$

$$|G(s) \cdot H(s)| = 1$$

$$\underline{/G(s) \cdot H(s)} = (2q+1)\ 180°$$

* The stability of a close loop control system is determine from the location of the roots of the characteristics equation.
* The system to be stable the roots of the characteristics equation must be located on the left side of the 'S' plane.
* The characterstice eqn $1 + G(s) \cdot H(s) = 0$
* The roots of the characteristics eqn must satisfy the below mentioned eqn

$$|G(s) \cdot H(s)| = 1$$

$$\underline{/G(s) \cdot H(s)} = (2q+1)\ 180°$$

Root locus:-

* The root locus method of analysis is a process of determining the ~~tt~~ point in 'S'- plane satisfying the magnitude and phase angle equation.
* Generally forward path gain factor (K) is considered as an indipendent variable and the roots (s) of the characteristic eqn will dépendent variable in this graph

Let $G(s) \cdot H(s) = \frac{ks}{s+1}$

$$1 + G(s) \cdot H(s) = 0$$

$$\Rightarrow 1 + \frac{ks}{s+1} = 0$$

$$\Rightarrow ks + s + 1 = 0$$

$$\Rightarrow k = \frac{-(s+1)}{s}$$

$$\Rightarrow k = \frac{-s}{s} - \frac{1}{s}$$

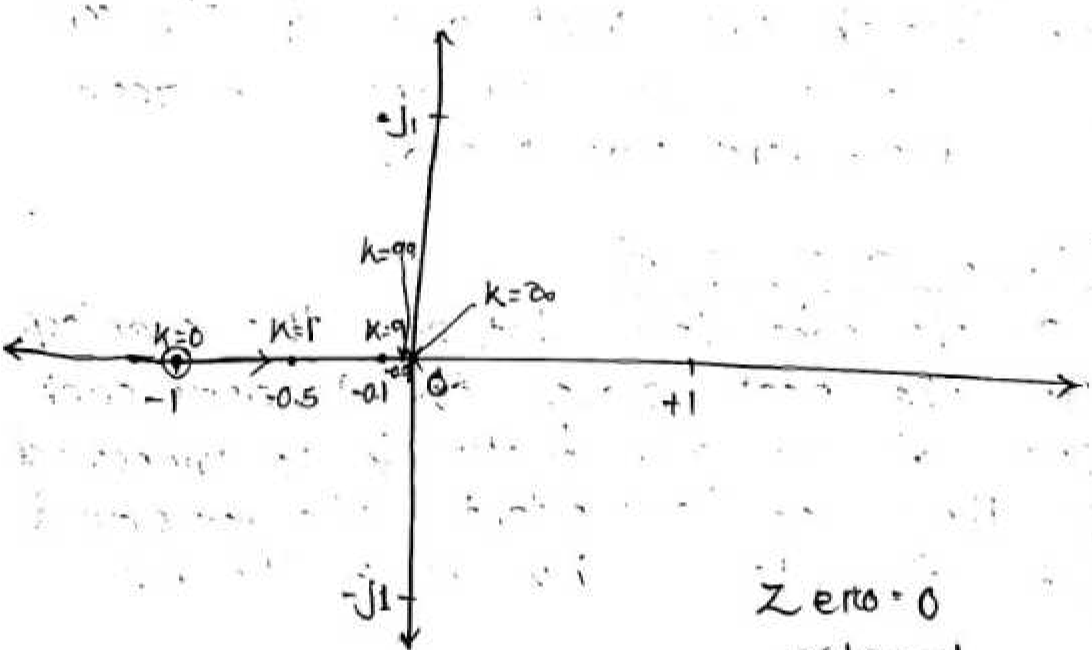
$$= -1 - \frac{1}{s}$$

$$\Rightarrow -\frac{1}{s} = k + 1$$

$$\Rightarrow \frac{1}{s} = 1 - k$$

$$\Rightarrow s = \frac{-1}{1+k}$$

$k =$	0	1	9	99	∞
$s =$	-1	-0.5	-0.1	-0.01	0



Zero = 0
pole = -1

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Procedure for ploting the root locous.

The root locus plopt can be drawn from a open loop transfer function G(s). H(s) obeying the following procedure.

1) Starting point.

The root locus starts from open loop poles.

2) Ending point:

The rot locus ends at $k = \infty$ i.e open loop zero or infinity.

3) No of root locus branches (N)

$N = P \quad P > Z$

$N = Z \quad Z < P$

4) Existance of root locus branch on the real axis.

* A section of the root locus branch will pass through the real axis if the same sum of no. of ~~p~~ open loop poles and zero to the right hand side is odd.

5) Break away point.

* On the root locus between two open loop poles the roots moves ~~tor~~ towards each other as the gain factor k is increased till they are coincident, the coincident point is obtained by solving the eqn

$$\frac{dk}{ds} = 0$$

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* Any further increase in the values of k' the root locus breaks away into two parts. that can be obtained from the above equation.

c) The angle of passing poles asymptode

For heigher values of 'k' the root locus branches are approximated by assymtoting asymptotic lines.

$$\frac{(2q+1)180^\circ}{p-z} \qquad q = 0, 1 \ldots p-z-1$$

7) The asymtode intersect at a point x on the real axis when

$$x = \frac{\Sigma \text{ poles} - \Sigma \text{ zeros}}{p-z}$$

8) Intersection points with imaginary axis

* The value of k and the point at which the root locus branch crosses the imaginary axis is determine by appling routh criteria to the characteristics equation.
* The roots at the intersection points are imaginary.

9) The angle of departure from complex pole

$$\phi_d = 180^\circ - (\phi_p - \phi_z)$$

ϕ_p = Sum of angle of depart subtended by remaining poles.

ϕ_z = sum of angle subtended by all zeros.

* The angle of departure is tangent to the root locus at complex plane.

10) The angle of arival at complex zero

$$\phi_a = 180^\circ (\phi_z - \phi_p)$$

ϕ_z: Sum of angle subtended by remaining zero

ϕ_p: Sum of angle of subtended by all poles.

* The angle of arival is tangent to root locos at complex zero.

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Prob

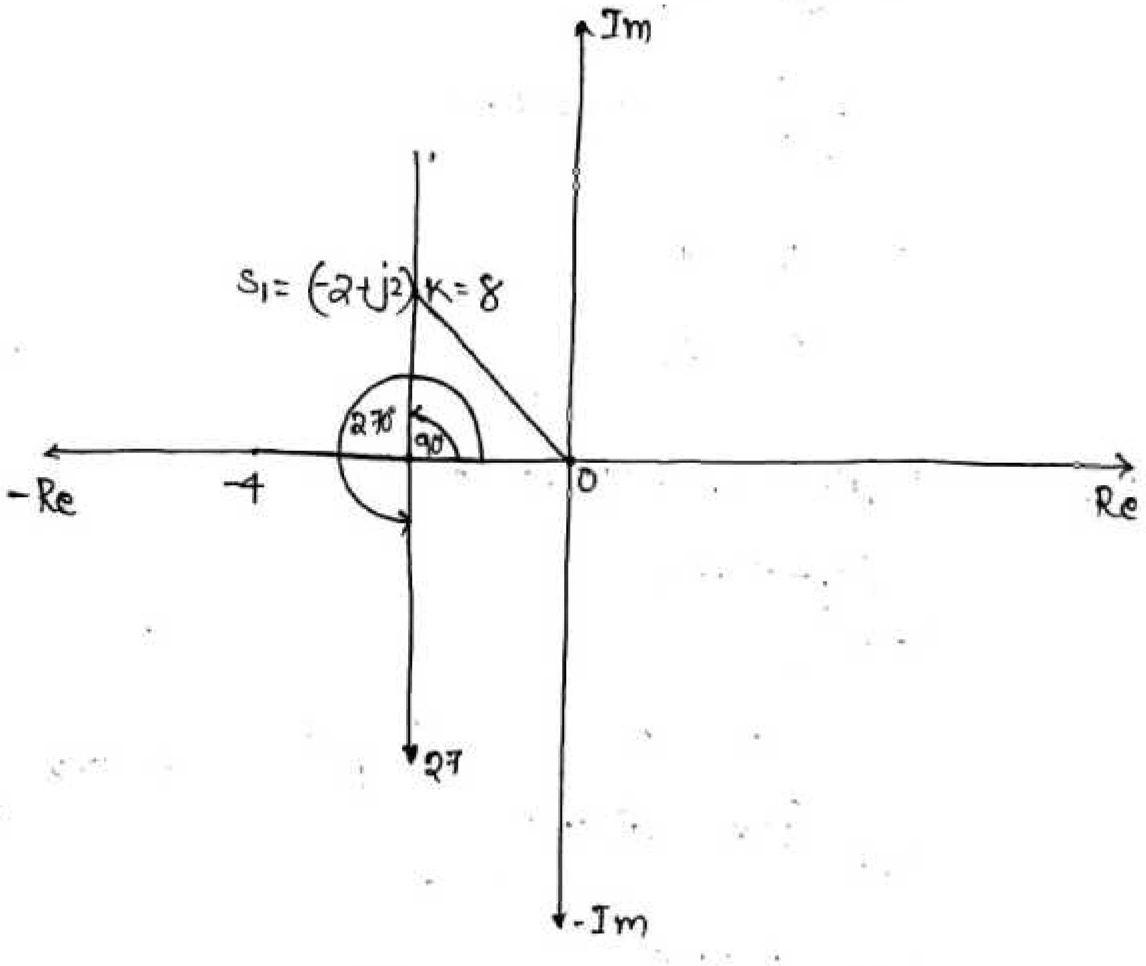
$$G(s) = \frac{k}{s(s+4} \qquad H(s) = 1$$

$$K = ? \qquad \xi = 0.707 = \frac{1}{\sqrt{2}}$$

$$G(s) \cdot H(s) = \frac{k}{s(s+4)} \times 1 = \frac{k}{s(s+4)}$$

open loop poles $P = 0, -4$

open loop zeros $Z = Nil$



Step-1 : Starting point $s = 0, -4$

Step-2 : Ending points : infinity.

Step-3 : $N = P$, $P > Z$ $\qquad P = 2 \quad Z = 0$

$N = 2$

Step-4: The root locus branch must lie within $-4, 0$.

Step-5: Break away point

$$\frac{dk}{ds} = 0$$

$$1 + G(s).H(s) = 0$$

$$\Rightarrow 1 + \frac{k}{s(s+4)} = 0$$

$$\Rightarrow s^2 + 4s + k = 0$$

$$\Rightarrow k = -s^2 - 4s$$

$$\Rightarrow \frac{dk}{ds} = -2s - 4 = 0$$

$$\Rightarrow -2s = 4$$

$$\Rightarrow s = -2.$$

Step-6 - Angle of asymptode.

$$\phi = \frac{(2q+1)180^\circ}{p-z}$$

$q = 0, 1, 2, \ldots, p-z-1$

$p = 2, \quad z = 0, \quad p-z = 2$

$$\phi_1 = \frac{(2\times 0 + 1)180^\circ}{2} = 90^\circ$$

$$\phi_2 = \frac{2\times 1 + 1}{2} \times 180^\circ = 270$$

$$\cos\theta = \xi$$

$$\theta = \cos^{-1}\xi$$

$$= 45^\circ$$

$$|G(s).H(s)| = 1$$

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$$S_1 = -2, 2 = -2 + j2$$

$$\left| \frac{K}{s_1(s_1+4)} \right| = 1$$

$$\left| \frac{K}{(-2+j2)(-2+j2+4)} \right| = 1$$

$$\left| \frac{K}{(-2+j2)\ (2+j2)} \right| = 1$$

$$\left| \frac{K}{-4 - \cancel{j4} + \cancel{j4} - 4} \right| = 1$$

$$\left| \frac{K}{-8} \right| = 1$$

$$K = 8$$

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Frequency response Analysis

The magnitude and phase relationship betw^n the sinusoidal input and steady state output of a system is termed as frequency response.

POLAR PLOT

The sinusoidal transfer function $G(j\omega)$ is a complex function is given by.

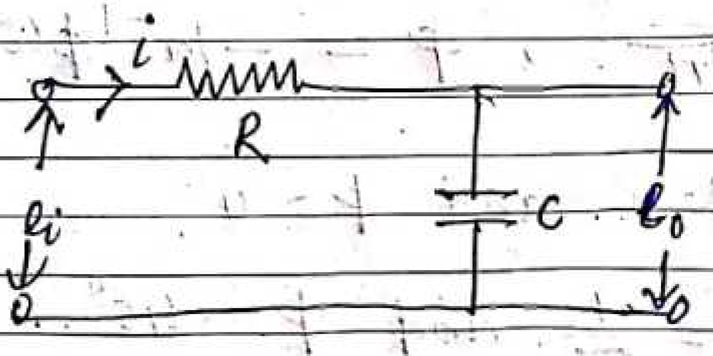
$$G(j\omega) = Re[G(j\omega)] + j\,Im[G(j\omega)]$$

$$G(j\omega) = |G(j\omega)| \angle G(j\omega) = M \angle \phi$$

$G(j\omega)$ may be represented as a phasor magnitude M and phase angle ϕ

As ω, the input frequency varied from 0 to ∞, the magnitude M and phase angle ϕ changes. hence the tip of the phasor $G(j\omega)$ traces a locus in the complex plane. The locus thus obtained is called polar plot

Consider a R-C Filter.



$$e_0 = i X_C = \frac{i}{\omega C}$$

$$E_0(s) = \frac{I(s)}{sC} \qquad s = j\omega.$$

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$$e_i = iR + iX_C = i\left(R + \frac{1}{\omega C}\right)$$

$$E_i(s) = I(s)\left(R + \frac{1}{sC}\right)$$

$$G(s) = \frac{E_0(s)}{E_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + RCs}$$

Where $T = RC$

$$G(s) = \frac{1}{1 + Ts}$$

$$G(j\omega) = \frac{1}{1 + j\omega T}$$

$$\cancel{G(j\omega) = \frac{1}{R+}}$$

$$G(j\omega) = \frac{1}{\sqrt{1 + \omega^2 T^2}} \angle -\tan^{-1} \omega T$$

$$= M \angle \phi$$

Where $M = \frac{1}{\sqrt{1 + \omega^2 T^2}}$, $\phi = -\tan^{-1} \omega T$.

When $\omega = 0$, $M = 1$ and $\phi = 0$.

$\omega = \frac{1}{T}$ $M = \frac{1}{\sqrt{2}}$ and $\phi = -45^\circ$.

$\omega \to \infty$ $M = 0$ and $\phi = -90^\circ$

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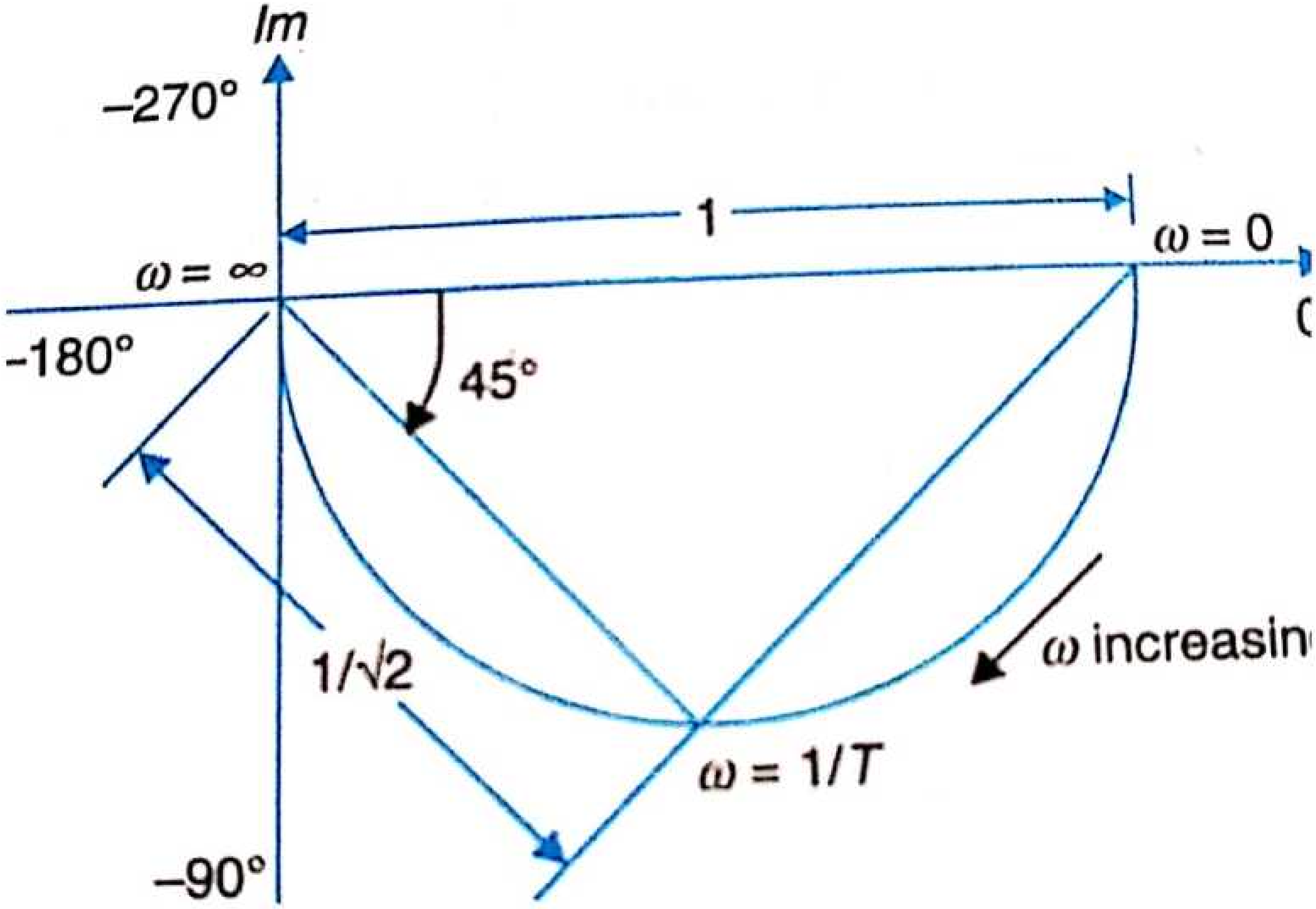

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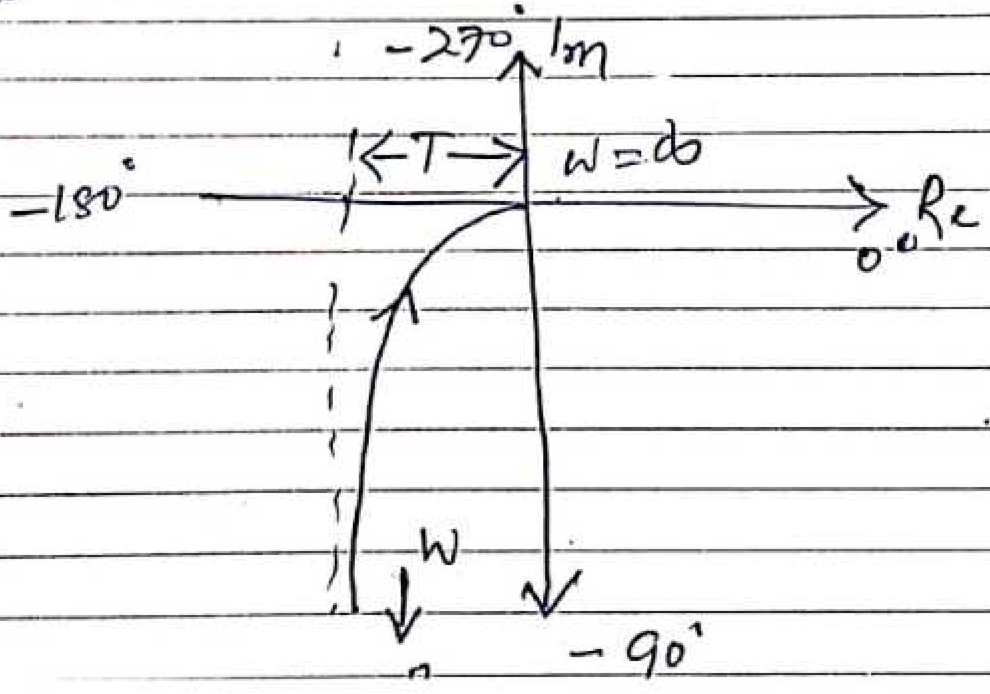
Consider another transfer function.

$$G(j\omega) = \frac{1}{j\omega\,(1+j\omega T)}$$

$$G(j\omega) = \frac{-T}{(1+\omega^2T^2)} - j\,\frac{1}{\omega(1+\omega^2T^2)}$$

$$\lim_{\omega \to 0} G(j\omega) = -T - j\infty = \infty \angle -90^\circ$$

$$\lim_{\omega \to \infty} G(j\omega) = -0 - j0 = 0 \angle -180^\circ$$



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This transfer function may be rearranged as

$$G(j\omega) = \frac{-T}{1+\omega^2T^2} - j\frac{1}{\omega(1+\omega^2T^2)} \qquad ...(8.14)$$

From eqn. (8.14) we get

$$\lim_{\omega \to 0} G(j\omega) = -T - j\infty = \infty\angle -90°$$

$$\lim_{\omega \to 0} G(j\omega) = -0 - j0 = 0\angle -180°$$

The general shape of the polar plot of this transfer function is shown in Fig. 8.8. The plot is asymptotic to the vertical line passing through the point $(-T, 0)$.

The major advantage of the polar plot lies in stability study of systems. N. Nyquist (in 1932) related the stability of a system to the form of these plots. Because of his work, the polar plots are commonly referred to as *Nyquist plots*.

The general shapes of the polar plots of some important transfer functions are given in Table 8.1.

From the polar plots of Table 8.1, following observations are made:

(*i*) Addition of a nonzero pole to a transfer function results in further rotation of the polar plot through an angle of $-90°$ as $\omega \to \infty$.

(*ii*) Addition of a pole at the origin to a transfer function rotates the polar plot at zero and infinite frequencies by a further angle of $-90°$.

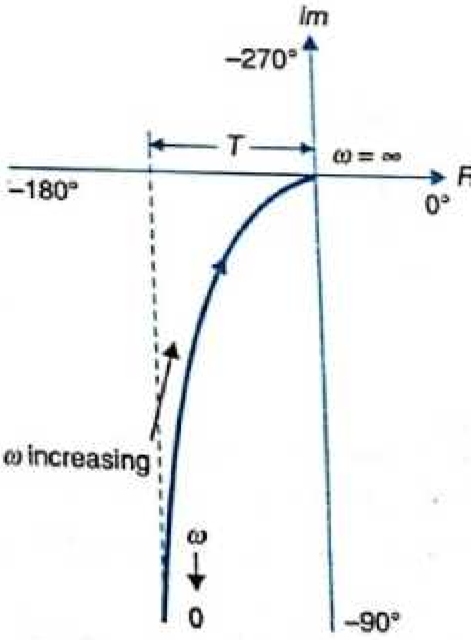
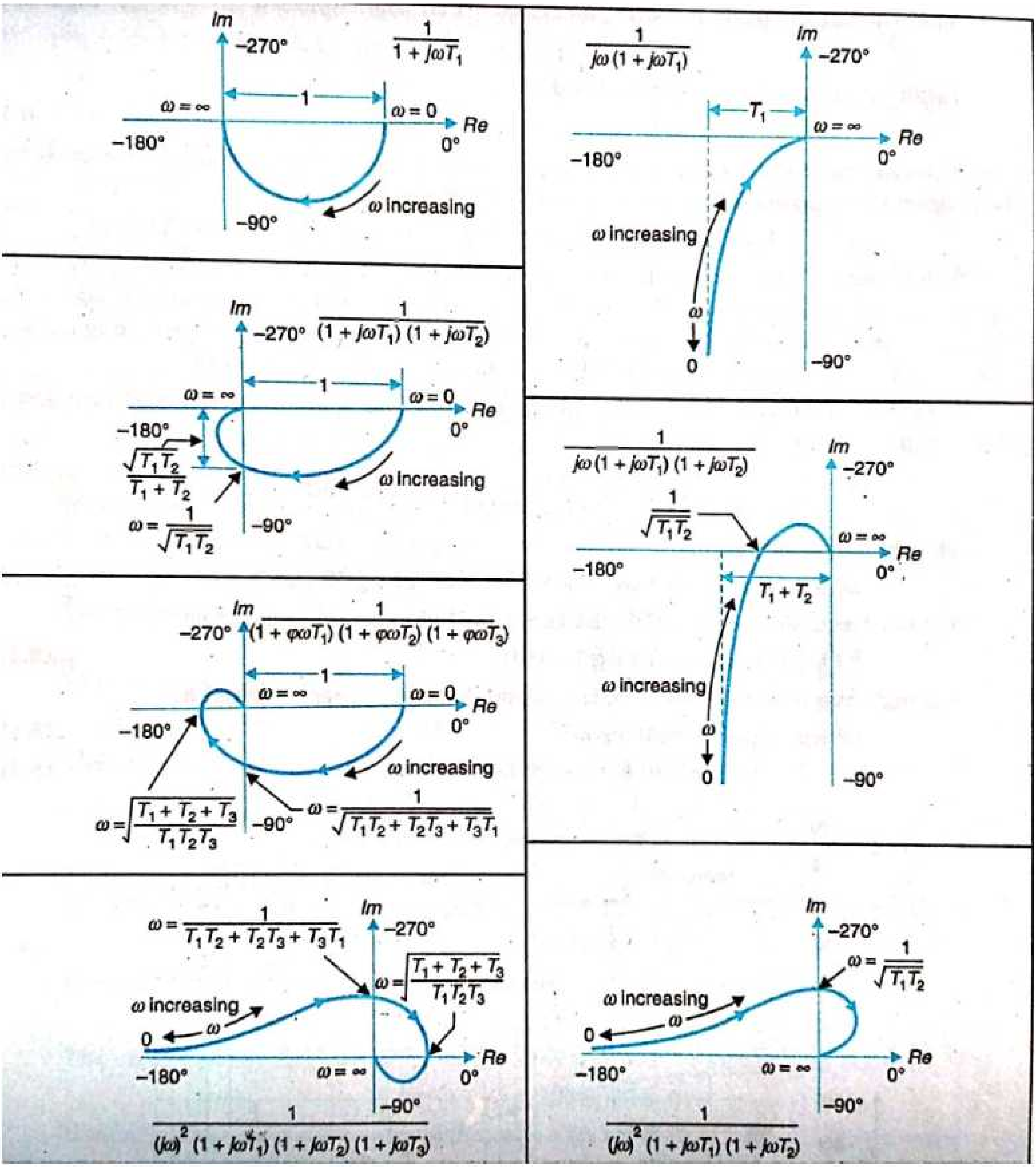


Fig. 8.8. Polar plot of $1/j\omega(1 + j\omega T)$

The effect of addition of a zero to a transfer function is to rotate the high frequency portion of the polar plot by 90° in counter-clockwise direction.



$\frac{1}{1 + j\omega T_1}$
Im
−270°
1
ω = ∞
ω = 0
Re
−180°
0°
ω increasing
−90°
$\frac{1}{j\omega(1 + j\omega T_1)}$
Im
−270°
T_1
ω = ∞
Re
−180°
0°
ω increasing
ω
0
−90°
$\frac{1}{(1 + j\omega T_1)(1 + j\omega T_2)}$
Im
−270°
1
ω = ∞
ω = 0
Re
−180°
0°
$\frac{\sqrt{T_1 T_2}}{T_1 + T_2}$
ω increasing
$\omega = \frac{1}{\sqrt{T_1 T_2}}$
−90°
$\frac{1}{j\omega(1 + j\omega T_1)(1 + j\omega T_2)}$
Im
−270°
$\frac{1}{\sqrt{T_1 T_2}}$
ω = ∞
Re
−180°
0°
$T_1 + T_2$
ω increasing
ω
0
−90°
$\frac{1}{(1 + \varphi\omega T_1)(1 + \varphi\omega T_2)(1 + \varphi\omega T_3)}$
Im
−270°
1
ω = ∞
ω = 0
Re
−180°
0°
ω increasing
$\omega = \sqrt{\frac{T_1 + T_2 + T_3}{T_1 T_2 T_3}}$
−90°
$\omega = \frac{1}{\sqrt{T_1 T_2 + T_2 T_3 + T_3 T_1}}$
$\omega = \frac{1}{T_1 T_2 + T_2 T_3 + T_3 T_1}$
Im
−270°
$\omega = \sqrt{\frac{T_1 + T_2 + T_3}{T_1 T_2 T_3}}$
ω increasing
ω
0
Re
−180°
ω = ∞
0°
−90°
$\frac{1}{(j\omega)^2 (1 + j\omega T_1)(1 + j\omega T_2)(1 + j\omega T_3)}$
Im
−270°
$\omega = \frac{1}{\sqrt{T_1 T_2}}$
ω increasing
ω
0
Re
−180°
ω = ∞
0°
−90°
$\frac{1}{(j\omega)^2 (1 + j\omega T_1)(1 + j\omega T_2)}$

BODE PLOT

One of the most useful representation of a transfer function is a logarithmic plot.

It consists of two graphs.

(1) The Logarithmic of $|G(j\omega)|$

(2) phase angle . both plotted,

versus

frequency in logarithmic scale.

These plots are called Bode plots in honour of H. W. Bode

Or

The variation of the magnitude of sinusoidal transfer function expressed in decibel and corresponding phase angle in degree being plotted w.r.t frequency on a logarithmic scale (i.e $\log_{10} \omega$) in rectangular axes. The plot thus obtained is known as Bode plot.

$$G(j\omega) = |G(j\omega)| e^{j\phi}. \quad —(1)$$

Taking natural logarithmic of both sides

$$\ln G(j\omega) = \underbrace{\ln|G(j\omega)|}_{Re} + j\underbrace{\phi(\omega)}_{Im} \quad —(2)$$

Real part is the natural logarithmic of magni and is measured in a basic unit called nepe

The imaginary part is the phase characte

Taking logarithmic of base 10 on both side of equⁿ (1).

$$\log G(j\omega) = \log|G(j\omega)| + \log e^{j\phi(\omega)}$$

$$= \log|G(j\omega)| + j\phi(\omega)\log_{10} e$$

$$= \log|G(j\omega)| + j0.434\,\phi(\omega) \quad (3)$$

$20\log|G(j\omega)|$ and phase angle $\phi(\omega)$ versus $\log\omega$.

Unit of Magnitude $20\log|G(j\omega)|$ is decibel abbreviated as db.

The curve generally drawn on semilog paper using log scale for frequency and linear scale for magnitude in db and phase angle in degrees.

Consider an example RC Filter.

Sunday 10

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$

The Log-magnitude is

$$20\log|G(j\omega)| = 20\log(1+\omega^2T^2)^{-\frac{1}{2}}$$

$$= -10\log(1+\omega^2T^2) \quad (4)$$

For low frequency $\omega \ll \frac{1}{T}$

$$20\log|G(j\omega)| = -10\log 1 = 0\text{ db} \quad (5)$$

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S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

For High frequency $\omega \gg \frac{1}{T}$

$$20\log|G(j\omega)| = -20\log\omega T$$

$$= -20\log\omega - 20\log T \quad (6)$$

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The logarithmic ~~scale~~ plot $20 \log |G(j\omega)| \sim \log \omega$ of eqn (5) is a horizontal axis.

The plot of eqn (6) is a straight line with a slope -20 db per unit change in log

A unit change $\log \omega$ means.

$$\log\left(\frac{\omega_2}{\omega_1}\right) = 1 \qquad \omega_2 = 10\,\omega_1$$

The range of frequency is called decade
Thus the slope -20 db/decade.

The range of frequency $\omega_2 = 2\omega_1$ is called oct

$$-20 \log 2 = -6 \text{db}$$

Slope is called -6 db/octave.

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Though the straight line approximations of eqn. (8.17) and (8.19) hold good for $\omega << 1/T$ and $\omega >> 1/T$ respectively, with some loss of accuracy these could be extended for frequencie $\omega \leq 1/T$ and $\omega \geq 1/T$. Therefore the log-magnitude versus log-frequency curve of $1/(1 + j\omega T)$ ca be approximated by two straight line asymptotes, one a straight line at 0 db for the frequenc range $0 < \omega \leq 1/T$ and the other, a straight line with a slope –20 db/decade (or – 6db/octave) f the frequency range $1/T \leq \omega < \infty$. The frequency $\omega = 1/T$ at which the two asymptotes meet called the *corner frequency* or the *break frequency*. The corner frequency divides the plot in t regions, a low frequency region and a high frequency region.

It is important to note that the log-magnitude plot of $(1 + j\omega T)^{-1}$ shown in Fig. 8.10 is asymptotic approximation of the actual plot. The actual plot can be obtained from it by apply correction for the errors introduced by asymptotic approximation.

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The error in logmagnitude for $0 < \omega \leq \frac{1}{T}$

$$-10 \log(1+\omega^2T^2) + 10 \log 1$$

Error at corner frequency, $\omega = \frac{1}{T}$ is

$$-10 \log(1+1) + 10 \log 1 = -3\,db$$

For $\frac{1}{T} \leq \omega < \infty$, the error in log magni

$$-10 \log(1+\omega^2T^2) + 20 \log \omega T$$

Error at corner frequ .

2012

T	F	S
3	1	2
10	8	9
17	15	16
24	22	23
31	29	30

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$\omega = \frac{1}{T}$

$-10\log(1+1) + 20\log 1 = -3\,db.$

Bode plot (Logarithmic plot) for Transfer function

$$G(s) = \frac{K[(1+sT_1)(1+sT_2)\ldots]\omega_n^2}{s^N(1+sT_a)(1+sT_b)\ldots(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$G(j\omega) = \frac{K[(1+j\omega T_1)(1+j\omega T_2)\ldots]\omega_n^2}{(j\omega)^N(1+j\omega T_a)(1+j\omega T_b)\ldots(\omega_n^2-\omega^2+j2\xi\omega_n\omega)} \quad -(7)$$

The procedure for drawing the Bode plot for Transfer function

In decibel.

$$20\log_{10}|G(j\omega)| = 20\log K + 20\log(1+j\omega T_1) + \ldots - 20N\log_{10}|j\omega| - 20\log(1+j\omega T_a) - 20\log_{10}\left|\frac{(\omega_n^2-\omega^2)}{\omega_n^2} + j\frac{2\xi\omega_n\omega}{\omega_n^2}\right| \quad -(8)$$

For Phase angle.

$$\angle G(j\omega) = \tan^{-1}\omega T_1 + \tan^{-1}\omega T_2 + \ldots - N\angle 90^\circ - \tan^{-1}\omega T_a - \tan^{-1}\omega T_b \ldots - \tan^{-1}\left[\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}\right] \quad (9)$$

The Bode plot is a graph obtained from equ^n (8) & (9) consisting of two parts.

(i) $20 \log_{10} |G(j\omega)| \sim \log_{10}\omega$

(ii) $\angle G(j\omega) \sim \log_{10}\omega$.

Graphs for the Gain Term K.

The magnitude in decibel for the term of K.

$$K(dB) = 20 \log_{10}(K) \quad \text{— (10)}$$

equ^n (10) indicates that the magnitude is independent of $\log_{10}\omega$ and as K is considered positive real.

fig.

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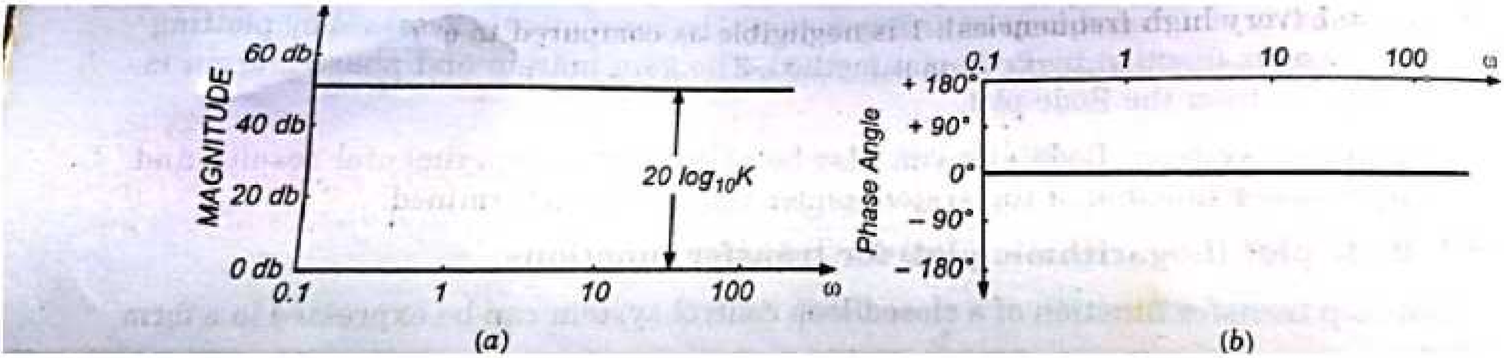

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7.18.3. Graphs for the Term $\frac{1}{(j\omega)^N}$

The magnitude of the term $1/(j\omega)^N$ in decibel is given by

$$20 \log_{10} \left| \frac{1}{(j\omega)^N} \right| = -20\, N \log_{10} \omega \quad ...(7.52)$$

The phase angle is given by $\angle \frac{1}{(j\omega)^N} = -90\, N°$ ...(7.53)

In view of Eqs. (7.52) and (7.53) the graphs are shown in Fig. 7.18.2. The graph for th magnitude versus $\log_{10} \omega$ is a straight line having a slope of – 20 N db/decade.

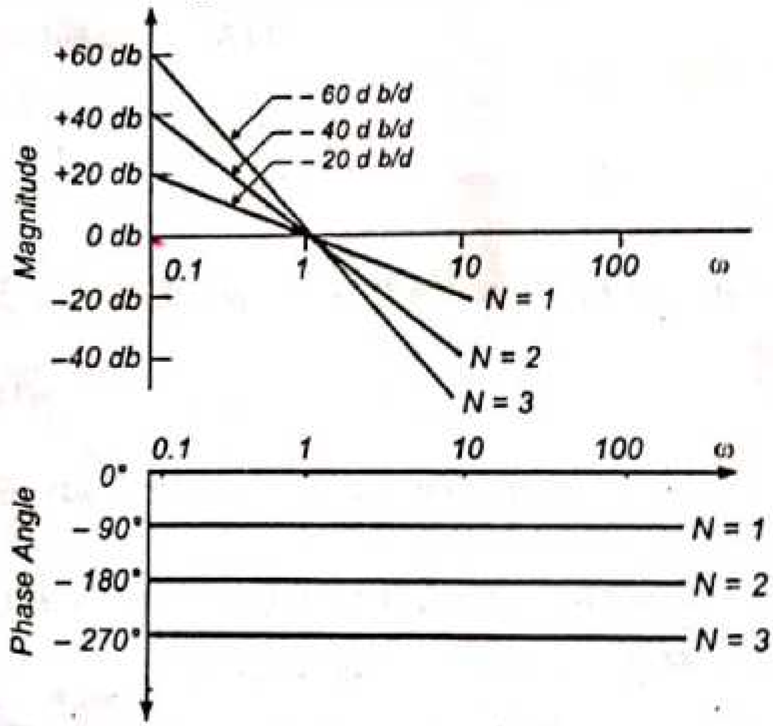

Fig. 7.18.2. Bode plot for the term $1/(j\omega)^N$.

As the term $1/(j\omega)^N$ has only imaginary term in the denominator the phase ang - 90 N°.

7.18.4. Graphs for the Term $(1 + j\omega T)$

The magnitude in decibel for the term $(1 + j\omega T)$ is given by

$$20 \log_{10} |(1 + j\omega T)| = 20 \log_{10} \sqrt{1 + \omega^2 T^2}$$

Consider following two cases :

(*i*) $\omega T \ll 1$ (very low frequencies), ωT is negligible as compared to 1.

$$\therefore \qquad 20 \log_{10} |(1 + j\omega T)| \simeq 20 \log_{10} 1 = 0 \text{ db}$$

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(ii) $\omega T \gg 1$ (very high frequencies), 1 is negligible as compared to ωT

$$20 \log_{10} |(1 + j\omega T)| \simeq 20 \log_{10} \sqrt{\omega^2 T^2} = 20 \log_{10} \omega T$$

$\therefore$

$$= 20 \log_{10} \omega - 20 \log_{10} (1/T) \quad ...(7.56)$$

In view of Eqs. (7.55) and (7.56) the graph for case (i) lies on 0 db axis, whereas for case (ii) the graph has a slope of 20 db/decade. These two graphs interest on 0 db axis at a point determined by equating the R.H.S. of Eq. (7.56) to zero.

$$0 = 20 \log_{10} \omega - 20 \log_{10} (1/T)$$

$\therefore$

$$20 \log_{10} \omega = 20 \log_{10} (1/T), \quad \therefore \quad \omega = (1/T)$$

or

Hence, the two graphs intersect on 0 db axis at $\omega = 1/T$.

The graphs for cases (i) and (ii) are shown in Fig. 7.18.3 (a).

The phase angle for the term $(1 + j\omega T)$ is given by

$$\phi = \tan^{-1}\left(\frac{\omega T}{1}\right)$$

(i) At very low frequencies ωT is very small

$\therefore$

$$\phi \simeq \tan^{-1}(0) \quad \text{or} \quad \phi = 0°$$

(ii) At

$$\omega = \frac{1}{T}$$

$$\phi = \tan^{-1}\left(\frac{1}{T} \cdot T\right) = \tan^{-1}(1)$$

or

$$\phi = 45°$$

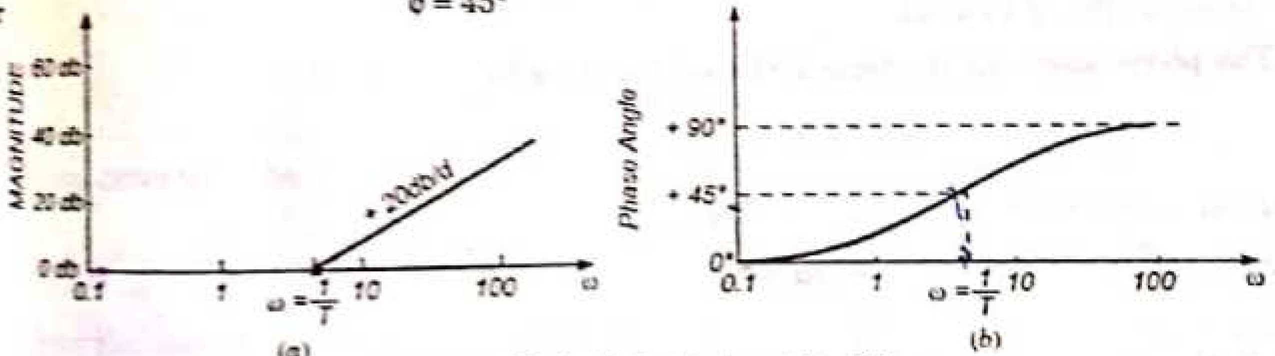

Fig. 7.18.3. Bode plot for the term $(1 + j\omega T)$.

(iii) At very high frequencies ωT is very large

$\therefore$

$$\phi \simeq \tan^{-1}(\infty)$$

or

$$\phi \simeq 90°$$

The graph for phase angle is shown in Fig. 7.18.3 (b).

7.18.5. Graphs for the term $\dfrac{1}{(1 + j\omega T)}$

The magnitude for the term $(1/(1 + j\omega T)$ in decibel is given by :

$$10 \log_{10} \left|\frac{1}{(1 + j\omega T)}\right| = 20 \log_{10} (1/\sqrt{1 + \omega^2 T^2})$$

$$= -20 \log_{10} \sqrt{1 + \omega^2 T^2} \quad ...(7.57)$$

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Graphs for the term $\left(\frac{1}{1+j\omega T}\right)$

The magnitude of $\frac{1}{1+j\omega T}$

$$20\log\left|\frac{1}{1+j\omega T}\right| = 20\log\frac{1}{\sqrt{1+\omega^2T^2}}$$

$$= -20\log\sqrt{1+\omega^2T^2}$$

(i) $\omega << \frac{1}{T}$, $\omega T << 1$

$$M = -20\log 1 = 0\ db \quad \text{--- (16)}$$

(ii) $\omega >> \frac{1}{T}$ $\quad \omega T >> 1$

$$M = -20\log \omega T$$

$$= -20\log\omega + 20\log\frac{1}{T} \quad \text{--- (17)}$$

Graph of Equ^n (16) & (17) meets the 0 db axis

$$20\log\omega = 20\log\frac{1}{T}$$

$$\omega = \frac{1}{T}$$

Hence the two graphs meets the 0 db axis at $\omega = \frac{1}{T}$.

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phase angle for the term $\frac{1}{1+j\omega T}$.

$$\phi = -\tan^{-1}\left(\frac{\omega T}{1}\right)$$

For Low frequeny $\phi = 0$.

for frequi $\omega = \frac{1}{T}$ $\phi = -45^\circ$.

For high frequeny $\phi = -90^\circ$

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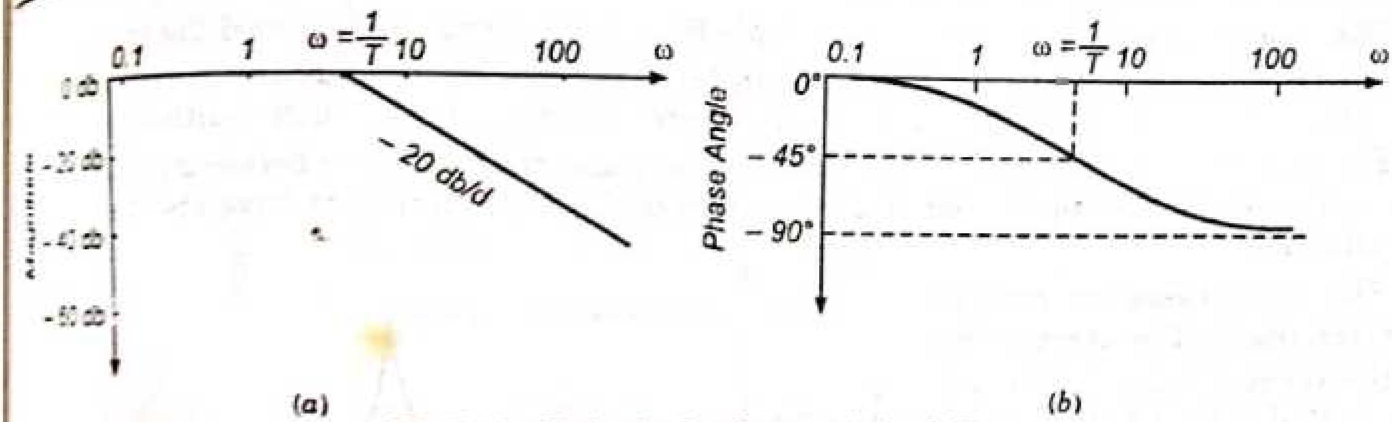

Fig. 7.18.4. Bode plot for the term $1/(1 + j\omega T)$.

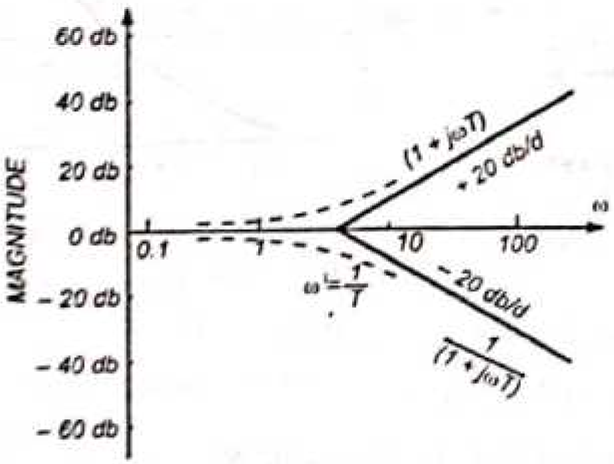

Fig. 7.18.5. Exact and asymptotic (approximate) bode plots for the terms $(1 + j\omega T)$ and $\frac{1}{(1 + j\omega T)}$

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INITIAL SLOPE OF BODE PLOT

The corner frequencies due to first order terms $(1+j\omega T_1)\,(1+j\omega T_2)\cdots\cdots \dfrac{1}{(1+j\omega T_a)}\,\dfrac{1}{(1+j\omega T_b)}\cdots$ etc.

are given by

$$\omega = \frac{1}{T_1}, \frac{1}{T_2} \cdots \frac{1}{T_a}, \frac{1}{T_b} \cdots \text{etc.}$$

For the frequencies lower than the lowest corner frequency the contribution towards gain of the transfer function is nil.

Transfer function for frequencies lower than the lowest corner frequencies can be expressed as,

$$G(j\omega) = \frac{K}{(j\omega)^N}.$$

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Magnitude

$$20 \log |G(j\omega)| = 20 \log \left| \frac{K}{(j\omega)^N} \right|$$

$$= 20 \log K - 20N \log \omega \quad \text{---(1)}$$

From this above equn. the graph magnitude vs. $\log_{10} \omega$ has initial slope of $-20N$ dB/decade.

N = Types of the transfer function.

<u>For Type (0) system i.e. N=0</u>

$$20 \log |G(j\omega)| = 20 \log_{10} \left| \frac{K}{(j\omega)^0} \right| = 20 \log_{10} K - 20 \log 1$$

$$= 20 \log_{10} K.$$

Initial slope for type (0) system is 0.

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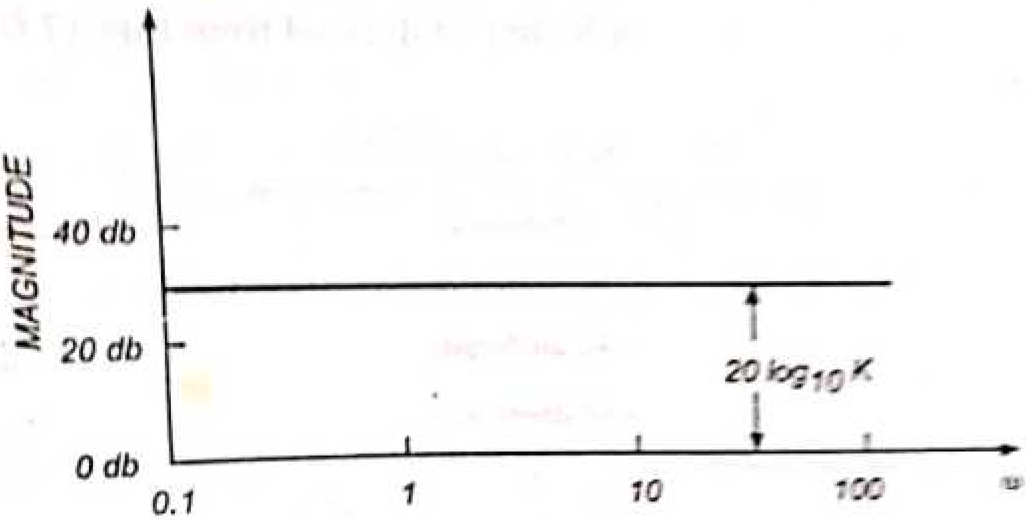

Fig. 7.18.7. Initial part of Bode plot for type 0 system

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V -

For Type '1' system i.e. N = 1

For type '1' system initial part of Bode plot is

$$20 \log_{10} |G(j\omega)| = 20 \log_{10} \left| \frac{K}{(j\omega)^1} \right|$$

$$= 20 \log_{10} K - 20 \log_{10} \omega$$

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Initial slope = -20 db/decade.

and the graph intersect the 0 db axis.

$$0 = 20\log_{10} K - 20\log_{10} \omega$$

$$20\log_{10} K = 20\log_{10} \omega$$

$$\omega = K$$

The graph intersect 0 db axis at $\omega = K$.

<u>For Type '2' system i.e $N=2$</u>

For type '2' system initial part of Bode plot.

$$20\log_{10} |G(j\omega)| = 20\log \left|\frac{K}{(j\omega)^2}\right|$$

$$= 20\log_{10} K - 20\log_{10} \omega^2$$

$$= 20\log_{10} K - 40\log_{10} \omega$$

Initial slope = -40 db/decade.
And the graph intersect the '0' db axis.

$$0 = 20\log_{10} K - 40\log_{10} \omega$$

$$20\log K = 20\log_{10} \omega^2$$

$$\omega^2 = K$$

$$\omega = \sqrt{K}$$

The graph intersect 0 db axis at $\omega = \sqrt{K}$

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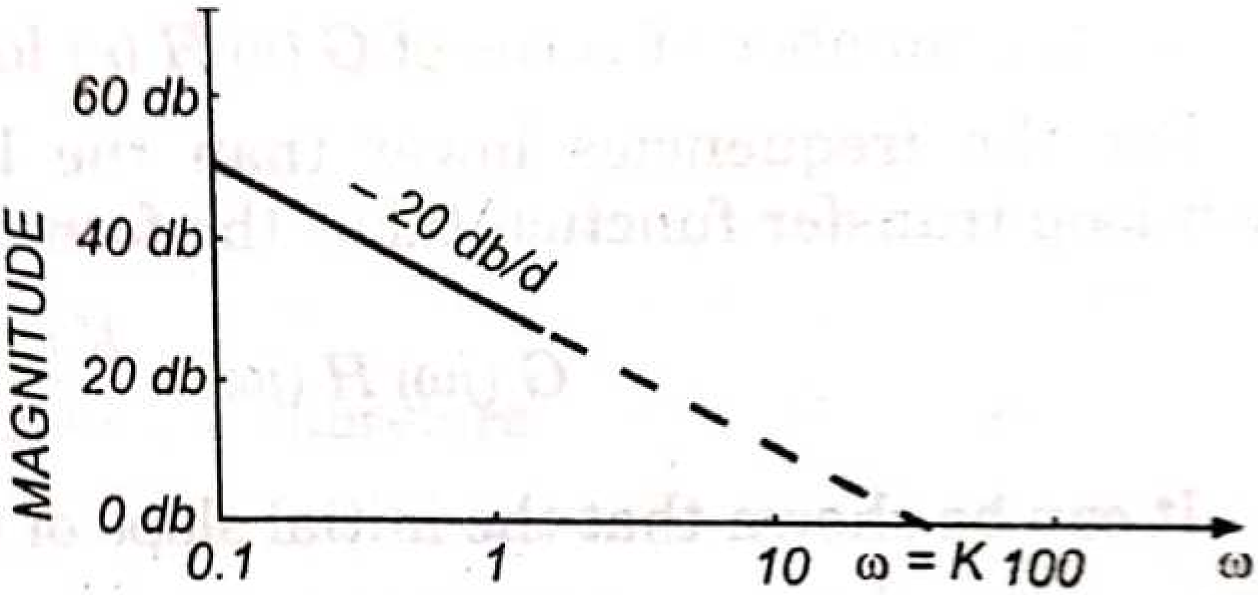

Fig. 7.18.8. Initial part of Bode plot for type 1 system.

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db/decade

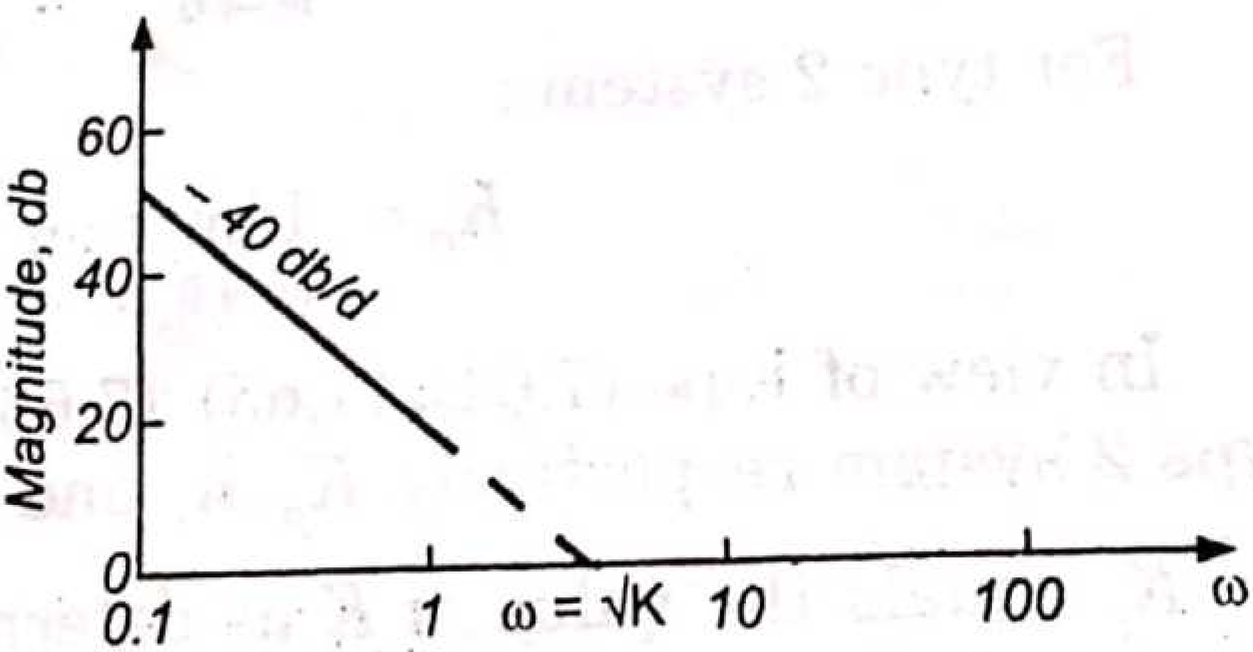

Fig. 7.18.9. Initial part of Bode plot for type 2 system.

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Similarly for type '3' system $N = 3$.

slope = -60 db/decade

intersection point on 0 db axis $\omega = \sqrt[3]{K} = K^{1/3}$

PROCEDURE FOR DRAWING BODE PLOT

Ex. Draw Bode plot for the system whose open loop T.f is

$$G(s) \cdot H(s) = \frac{4}{s(1+0.5s)(1+0.08s)}$$

$$G(j\omega)\,H(j\omega) = \frac{4}{j\omega\,(1+j0.5\omega)\,(1+j0.8\omega)}$$

1. The corner frequencies are

$\omega = \frac{1}{0.5} = 2$ rad/sec, and $\omega = \frac{1}{0.08} = 12.5$ rad/sec

2. Starting frequency is less than the lowest corner frequency.

As lowest corner frequency = 2 rad/sec

Starting frequency = 1 rad/sec.

3. As it is a Type '1' T.F.

Starting slope = -20 db/decade.

and intersection with '0' db axis

$\omega = 4$

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4. The denominator term $\frac{1}{(1+j0.5\omega)}$

Corner frequency = 2 rad/sec

slope = −20 db/decade.

Before slope was = −20 db/decade.

slope after ω = 2 rad/sec is −40 db/decade

= −20 db/decade + −20 db/decade

= −40 db/decade.

5. The denominator term $\frac{1}{(1+j\,0.8\omega)}$

Corner frequency ω = 12.5 rad/sec.

Slope due to this term = −20 db/decade.

Before slope was = −40 db/decade.

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Slope after ω = 12.5 rad/sec

Slope = −40 db/decade + −20 db/decade

= −60 db/decade.

This slope continues after ω = 12.5 rad/sec.

6. Phase angle $\angle G(j\omega)H(j\omega)$ for frequencies betw^n 1 rad/sec to 100 rad/sec.

ω (rad/sec)	1	2	8	10	20	50
$\angle G(j\omega)H(j\omega)$	-121	-144°	-198	-207	-234	-252

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ow .

$$\angle G(j\omega)\,H(j\omega) = -90^\circ - \tan^{-1}(0.5\omega) - \tan^{-1}(0.08\,\omega)$$

ω (rad/sec)	1	2	8	10	20	
$\angle G(j\omega)H(j\omega)^\circ$	− 121	− 144	− 198	− 207	− 234	−

The Bode plot $|G(j\omega)\,H(j\omega)|$ db and $\angle G(j\omega)\,H(j\omega)$ *versus* ω (log scale) is drawn an
Fig. 7.18.11.

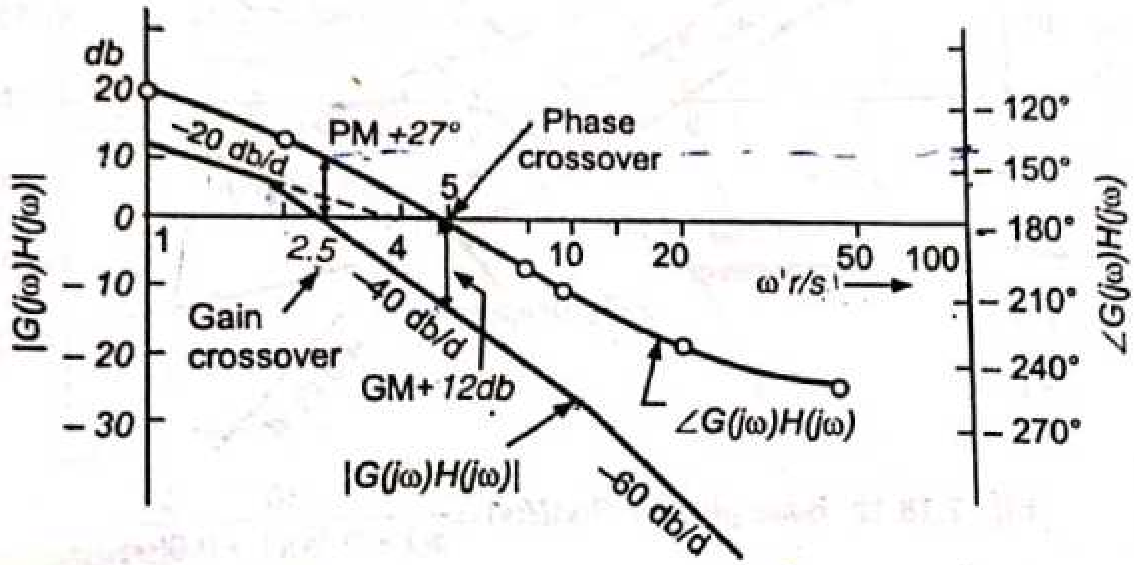

Fig. 7.18.11. Bode plot for $G(s)H(s) = \dfrac{4}{s(1+0.5s)(1+0.08s)}$.

Imp **Gain Margin:-**

The gain in db at phase cross over frequency is the gain margin. (G.M)

If Gain is $-ve$.
G.M is $+ve$.

Phase cross over frequency is 5 rad/sec.

and Gain $G(j\omega)\,H(j\omega) = -12\,db$

$G.M = +12\,db$.

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Phase Margin (P.M)

The phase margin is

$$P.M = 180° + \underline{|G(j\omega) \cdot H(j\omega)}$$

The gain crossover frequency is 2.5 rad/sec

$$\underline{|G(j\omega)H(j\omega)} = -153°$$

$$P.M = 180° + (-153°) = 27°$$

G.M & P.M both are +ve, Hence the closed loop system is stable.

For Stable systems:

The gain cross over frequency < phase crossover frequency.

For Unstable system:

The gain cross over frequency > phase cross over frequency.

For Marginally stable system

The gain cross over frequency = phase crossover freq

Gain Cross over frequency:

The frequency at which the Gain plot crosses the '0' db axis.

phase Cross over frequency:-

The frequency at which the ~~Gao~~ phase plot cross the '0' db axis.

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NYQUIST PLOT

PRINCIPLE OF ARGUMENT

Principle of argument states that if there are P poles and Z zeros are enclosed by the 's' plane closed path, then the corresponding G(s).H(s) plane must encircle the origin P-Z times.

Number of encirclements

$$N = P-Z.$$

If the enclosed 's' plane close path contains only poles, then the direction of the encirclement in the G(s).H(s) plane will be opposite to the direction of the closed path in the 's' plane.

If the enclosed 's' plane close path contains only zeros, then the direction of the encirclement in the G(s).H(s) plane will be in the same direction as that of enclosed path in the 's' plane.

Fig 9.3 Fig 9.4

NYQUIST STABILITY CRITERION.

Let us now apply the principle of argument to the entire right half of 's' plane by selecting it as a closed path.

This selecting path is called the Nyquist Contour.

We know that the closed loop control system is stable if all the poles of the closed loop transfer function are in the left half on 's' plane.

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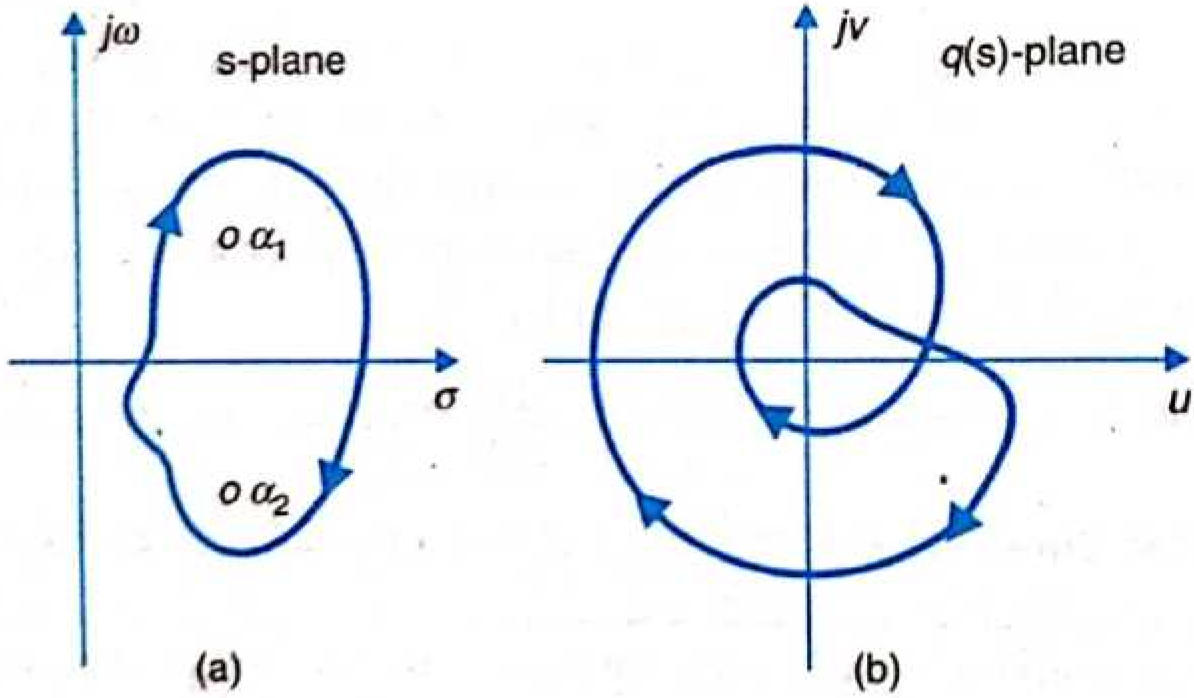

Fig. 9.3. An s-plane contour enclosing two zeros of q(s) and the corresponding q(s)-plane contour.

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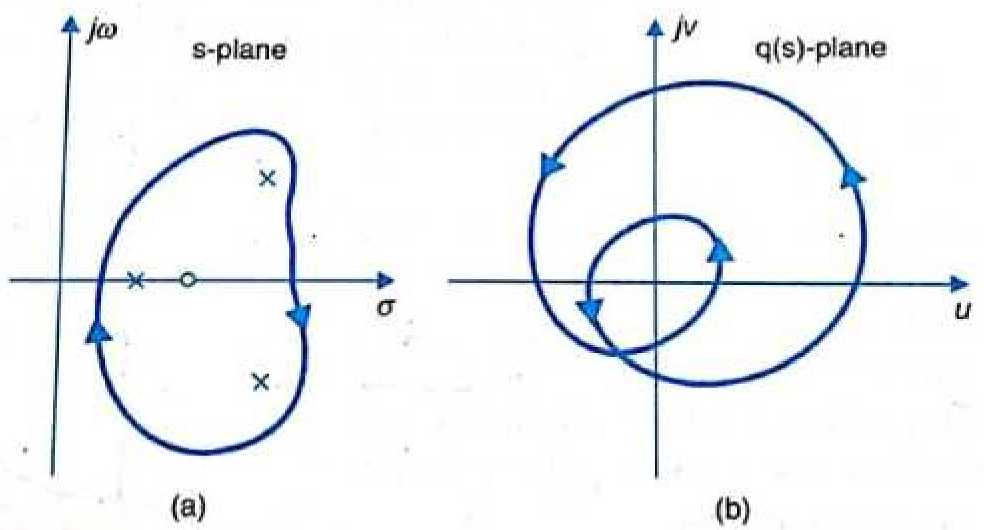

Fig. 9.4. Mapping of the s-plane contour which encloses 1 zero and 3 poles.

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poles of the closed loop transfer function are the nothing but the roots of the characteristic eqn.
i.e $1+G(s)\cdot H(s)=0$.
As the order of the characteristic eqn increases it is difficult to find the roots.

The poles of the characteristic eqn $(1+G(s)\cdot H(s)=0)$ are same as that of the poles of the open loop transfer function $(G(s)\cdot H(s))$.

The zeros of the characteristic eqn $(1+G(s)\cdot H(s)=0)$ are same as that of the zeros of the closed loop transfer function.

In order for the system to be stable there should be no zeros of $q(s)=1+G(s)H(s)$ on the right half s-plane.

$$Z=0$$

$$N=P$$

In special case of $P=0$ (i.e the open loop stable system) the closed loop system is stable if

$$N=P=0.$$

Which means the net encirclements of the origin of the q(s) plane by the Γ_q contour should be zero.

$$G(s)\cdot H(s)=[1+G(s)\cdot H(s)]-1$$

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24	28	29	30	31	29	30

Γ_{GH} contour of $G(s)H(s)$. Corresponding to Nyquist contour in the s plane is the same as contour Γ_q of $1+G(s)\cdot H(s)$ drawn for the point $-1+j0$.

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Encirclement of the origin by the contour Γ_q is equivalent to the encirclement of the point $(-1+j0)$ by the contour Γ_{GH}.

Fig 9.6. Fig 9.7

Along C_1

$s = j\omega$ with s varying from $-j\infty$ to $+j\infty$.

and along C_2

$s = \lim_{R \to \infty} Re^{j\theta}$ with θ varying from $+\pi/2$ to 0 to $-\pi/2$

Statement of Nyquist Stability criterion.

If the contour Γ_{GH} of the open loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane encircles the point $(-1+j0)$ in the counter clockwise dir^n as many as times as the number of right half s-plane poles of $G(s)H(s)$. the closed loop system is stable.

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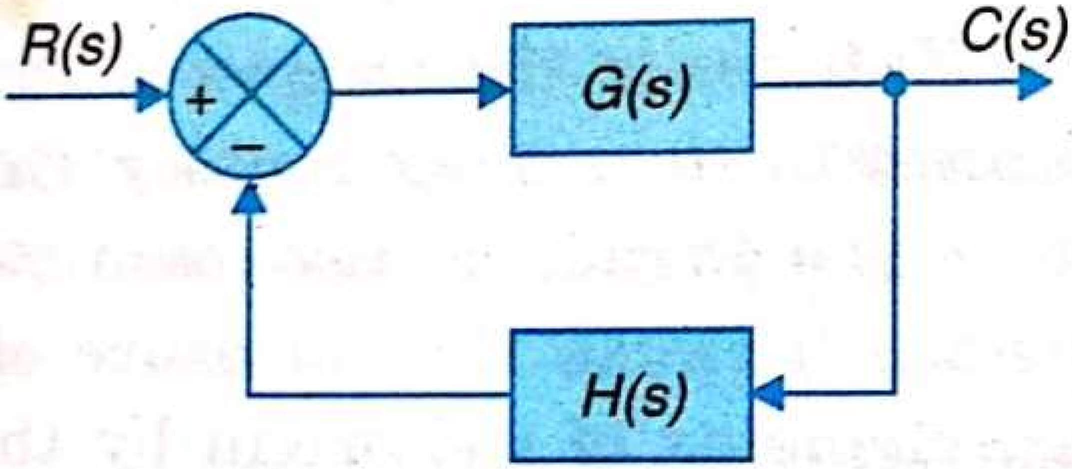

Fig. 9.5. A feedback control system.

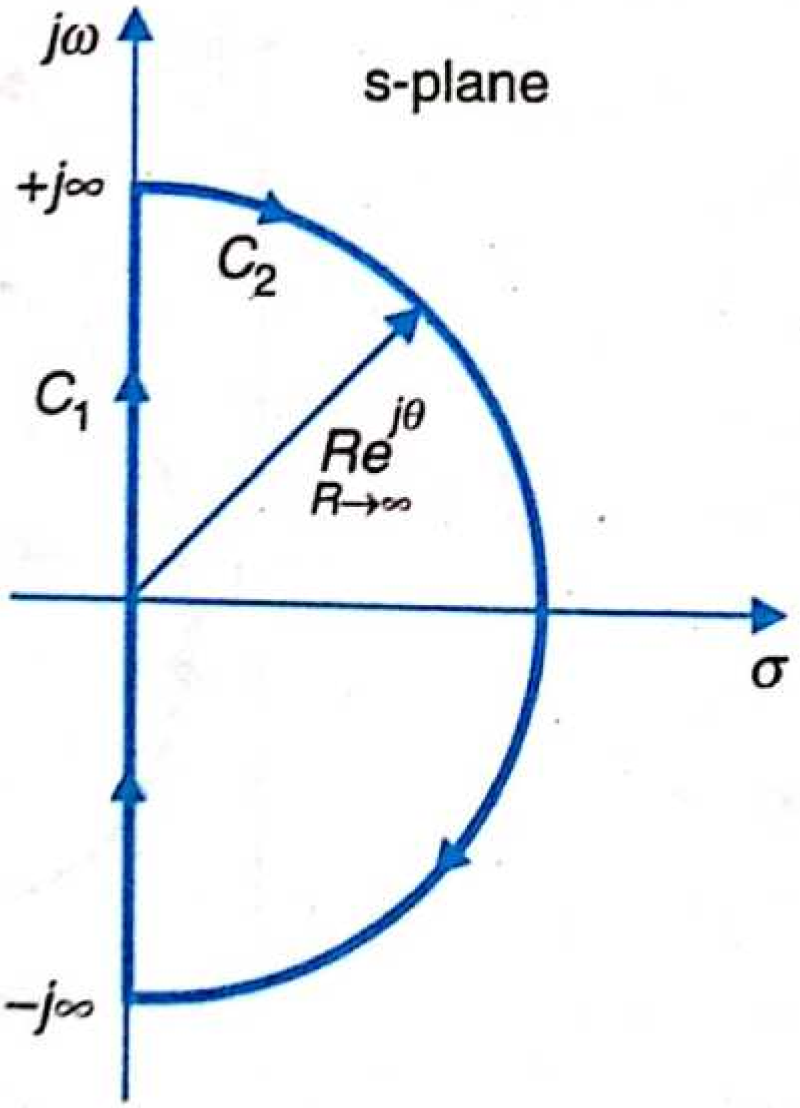

Fig. 9.6. The Nyquist contour.

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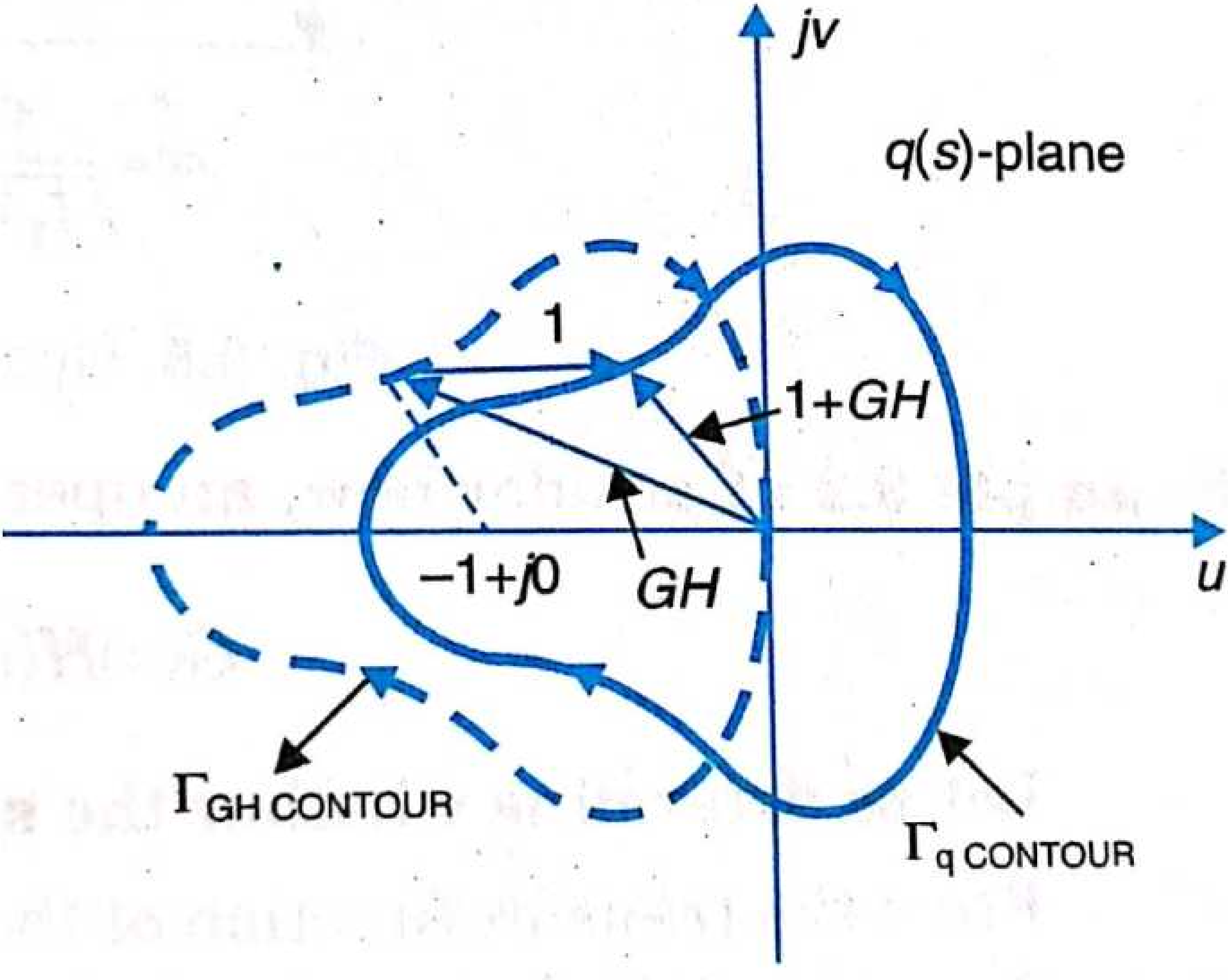

Fig. 9.7

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Ilustrative Example 1. *Examine the closed-loop stability of a control system whose open-loop transfer function is given below :*

$$G(s)\,H(s) = \frac{K}{s(sT + 1)}.$$

Solution.
$$G(s)\,H(s) = \frac{K}{s\,(sT + 1)}$$

Put
$$s = j\omega$$

$\therefore$
$$G(j\omega)\,H(j\omega) = \frac{K}{j\omega\,(j\omega T + 1)} \qquad ...(1)$$

Rationalizing Eq. (1) and separating into real and imaginary parts following equation is obtained

$$G(j\omega)\,H(j\omega) = -\frac{KT}{(\omega^2T^2 + 1)} - \frac{jK}{\omega\,(\omega^2T^2 + 1)} \qquad ...(2)$$

From Eq. (2) it is observed that as ω increases from $\omega = +0$ to $\omega = +\infty$ both the real part and the imaginary part lie in the third quadrant of $G(s)\,H(s)$-plane.

At $\omega = +0$, $\angle G(+j0)\,H(+j0) = -90°$ and the magnitude approaches infinity.

At $\omega = +\infty$, $\angle G(+j\infty)\,H(+j\infty) = -180°$ and the magnitude approaches zero.

The Nyquist plot as ω varied from $\omega = +0$ to $\omega = +\infty$ is shown in Fig. 7.6.7. The plot for $\omega = -\infty$ to $\omega = -0$ is shown by dotted lines.

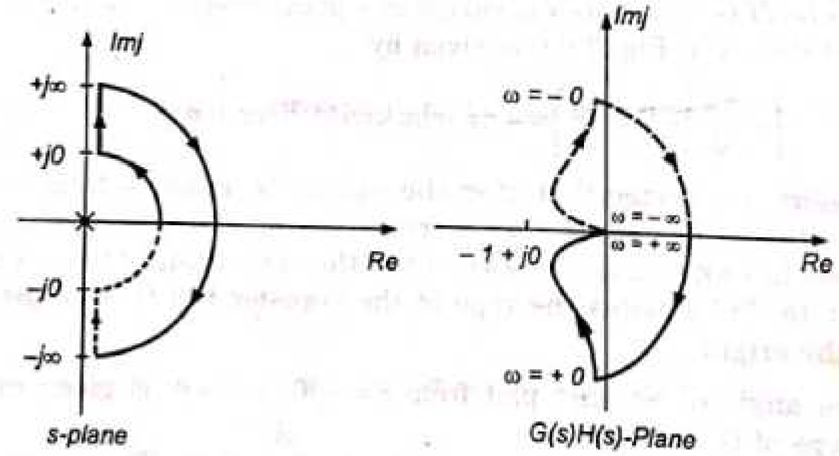

Fig. 7.6.7. Nyquist plot for $G(s)H(s) = \dfrac{K}{s(sT + 1)}$.

As the system is type 1 the plot is closed from $\omega = -0$ to $\omega = +0$ through an angle of $-\pi$ (clockwise) with an infinite radius. The arrowheads shown in Fig. 7.6.7 are in the direction of increasing ω.

As the point $(-1 + j0)$ is not encircled by the plot, therefore,

$$N = 0$$

The number of zeros (roots) of the characteristic equation with positive real part is determined by using relation

$$N = (P_+ - Z_+)$$

because, $\quad P_+ = 0 \qquad \therefore\ Z_+ = 0$

Hence, the number of zeros (roots of the characteristics equation) with positive real part is nil and the closed-loop system is stable.

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RULES FOR DRAWING NYQUIST PLOTS.

1) Locate the poles and zeros of the open loop transfer function $G(s).H(s)$. in s-plane.

2) Draw the polar plot varying ω from 0 to ∞. if the poles or zero present at $s=0$.

3) Draw the mirror image of above polar plot for values of ω ranging from $-\infty$ to 0^-.

4) The number of infinite radius half circles will be equal to the number of poles or zeros at origin.

The infinite radius half circle will start at the point where the mirror image of the polar plot ends.

And the infinite radius half circle will end at the point where the polar plot starts.

After drawing the Nyquist plot, we can find the stability of the closed loop control system using the Nyquist stability criterion.

If the critical point $(-1+j0)$ lies outside the encirclement, then the close loop control system is absolute stable.

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STABILITY ANALYSIS USING NYQUIST PLOT.

From the Nyquist plots, we can identify whether the control system is stable, marginally stable, unstable based on parameter,

(i) Gain cross over frequency (ω_{gc}) and phase cross over frequency (ω_{pc})

(ii) Gain Margin and phase Margin.

phase cross over frequency (ω_{pc})

The frequency at which the Nyquist plot intersects the negative real axis (phase angle is 180°) is known as phase cross over frequency (ω_{pc}).

Gain crossover frequency (ω_{gc})

The frequency at which the Nyquist plot is having the magnitude of one is known as ~~gain~~ gain cross over frequency (ω_{gc}).

For stable system $\omega_{pc} > \omega_{gc}$.

For Marginally stable system $\omega_{pc} = \omega_{gc}$.

For unstable system $\omega_{pc} < \omega_{gc}$.

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Gain Margin: (GM)

The gain margin GM is equal to the reciprocal of the Magnitude of the Nyquist plot at the phase cross over frequency.

$$G.M = \frac{1}{M_{pc}}$$

Where M_{pc} is the magnitude at phase cross over frequency in normal scale.

Phase Margin: (PM)

The phase margin (PM) is equal to the sum of 180° and the phase angle at the gain cross over frequency.

$$PM = 180^\circ + \phi_{gc}$$

Where ϕ_{gc} is the phase angle at gain cross over frequency

For stable system $GM > 1$, PM is +ve.

For marginally stable system $GM = 1$, $PM = 0$ degree

For unstable system $GM < 1$, PM is −ve.

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NYQUIST STABILITY CRITERION APPLIED TO INVERSE POLAR PLOT.

Occasionally, it is found more convenient to work with the inverse function $\frac{1}{G(j\omega)H(j\omega)}$ rather than the direct function $G(j\omega)H(j\omega)$.

Nyquist stability criterion can be applied to inverse polar plot, from ~~the~~ direct polar plot after minor modification.

Let us consider a open-loop transfer function.

$$G(s)H(s) = K\frac{(s+Z_1)(s+Z_2)\cdots(s+Z_m)}{(s+P_1)(s+P_2)\cdots(s+P_n)} ; \ m \le n. \quad -(1)$$

For stable system, no roots of the characteristic eqn should lie in the right half of s-plane.

$$q(s) = 1+G(s)H(s) = \frac{(s+Z_1')(s+Z_2')\cdots(s+Z_n')}{(s+P_1)(s+P_2)\cdots(s+P_n)} \quad -(2)$$

Dividing eqn (2) by eqn (1) we get

$$q'(s) = \frac{1}{G(s)H(s)} + 1 = \frac{(s+Z_1')(s+Z_2')\cdots(s+Z_n')}{(s+Z_1)(s+Z_2)\cdots(s+Z_m)} \quad -(3)$$

From eqn (2) & (3), it is found that

(i) Zeros of $q(s)$ & $q'(s)$ are same.

(ii) poles of $q(s)$ and $G(s)\cdot H(s)$ are same.

(iii) poles of $q'(s)$ and $\frac{1}{G(s)\cdot H(s)}$ are same.

and also same with zeros of $G(s)\cdot H(s)$.

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If $\frac{1}{G(s)H(s)}$ has P right half s-plane poles and characteristic eqn has Z right half s-plane zeros

The Locus $\frac{1}{G(s)H(s)}$ encircles the point $(-1+j0)$ N-times in counter clockwise dirn.

$$N = P - Z.$$

For stability $Z = 0$,

so, $N = P$.

If the Nyquist plot $\frac{1}{G(s) \cdot H(s)}$, corresponding to the Nyquist contour in the s-plane encircles $(-1+j0)$ in counter clockwise as many as the right half s-plane poles of $\frac{1}{G(s) \cdot H(s)}$.

Then the close loop system is stable.

Special case of no poles on right half s-plane of $\frac{1}{G(s)H(s)}$

$N = 0$, for stable system

Example 9.9 : Consider a feedback system with an open-loop transfer function

$$G(s)H(s) = K/s(Ts + 1)$$

The inverse polar plot of $G(s)H(s)$ corresponding to the s-plane Nyquist contour of Fig. 9.18(a) is obtained in steps below.

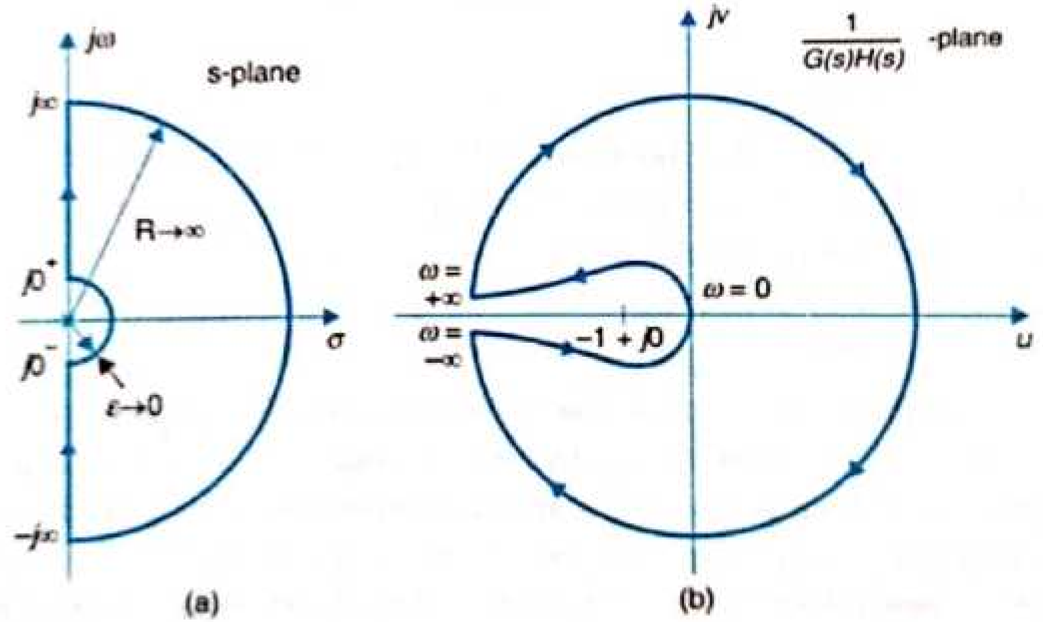

Fig. 9.18. The Nyquist contour and the corresponding plot of 1/G(s)H(s) = s(sT + 1)/K.

1. The semicircular indent around the origin in the s-plane is represented by

$$s = \lim_{\varepsilon \to 0} \varepsilon e^{j\theta}; \text{ where } \theta \text{ varies from } -90^\circ \text{ through } 0^\circ \text{ to } +90^\circ.$$

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It is mapped into $1/G(s)H(s)$-plane as

$$\lim_{\varepsilon \to 0} \left[\frac{\varepsilon e^{j\theta}(\varepsilon e^{j\theta} T + 1)}{K} \right] = \lim_{\varepsilon \to 0} \frac{\varepsilon}{K} e^{j\theta} = 0e^{j}\theta$$

2. Along the $j\omega$-axis $1/G(j\omega)H(j\omega) = j\omega(j\omega T + 1)/K$.

3. The infinite semicircle in the s-plane represented by

$$s = \lim_{R \to \infty} Re^{j\theta};\ \theta \text{ varies from } + 90° \text{ through } 0° \text{ to } -90°$$

is mapped into the $1/G(s)H(s)$-plane as

$$\lim_{R \to \infty} \frac{Re^{j\theta}(Re^{j\theta} + 1)}{K} = \lim_{R \to \infty} \frac{R^2}{K} e^{j2\theta}$$

which is a circle of infinite radius with angle varying from 180° through 0° to −180°.

The inverse polar plot of $G(s)H(s)$ obtained from the above steps is shown in Fig. 9.18(b). It is found that $(-1 + j0)$ point is not encircled by $1/G(s)H(s)$-locus. Further since $1/G(s)H(s) = s(Ts + 1)/K$ has no poles in the right half s-plane, the system is stable.

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7.8. RELATIVE STABILITY FROM NYQUIST PLOT

Fig. 7.8.1 shows Nyquist plots for two systems which are stable. As both the plots are crossing the negative real axis at same point, the two systems have the same gain margin. However, the two systems have different phase margin. The system B has more phase margin than system A. Therefore, system B is relatively more stable than system A.

Similarly, if two systems have same phase margin but different gain margin then the system having greater gain margin is relatively more stable than the system with lesser gain margin.

Conditionally stable systems. In the Nyquist plot shown in Fig. 7.8.2 the location of the point $(-1+j0)$ depends on the value of forward path gain K. For smaller range of K the point $(-1+j0)$ lies between oa, any increase in the value of K beyond this range brings the point $(-1+j0)$ between ab and if, K is further increased then the point $(-1+j0)$ lies between bc.

It is given that the open-loop transfer function of the system has the number of poles with positive real part as nil, therefore, if the point $(-1+j0)$ lies between oa or bc the number of encirclements of the point $(-1+j0)$ by the Nyquist plot are -2 indicating that the closed-loop system is unstable.

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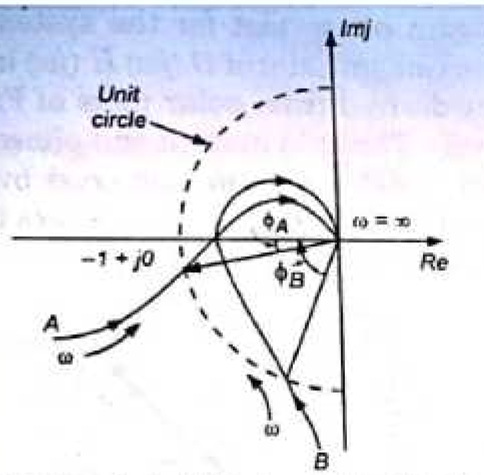

Fig. 7.8.1. Nyquist plot for two systems having same gain margin but different phase margin.

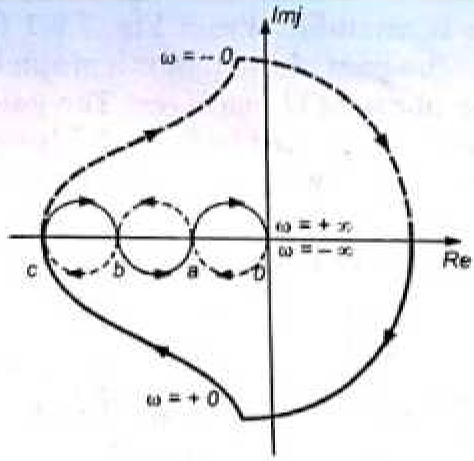

Fig. 7.8.2. Nyquist plot of conditionally stable system.

However, if the point $(-1 + j0)$ lies between ab the point $(-1 + j0)$ is encircled once in clockwise direction and then once again in the anti-clockwise direction resulting in net encirclements of the point $(-1 + j0)$ as nil indicating that the system is stable.

The system discussed above is stable only for a particular range of K any decrease or increase in the value of K makes the system unstable. Such systems are called conditionally stable systems.

The concepts of gain margin and phase margin are not applicable to conditionally stable systems.

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CONSTANT- M-CIRCLES (MAGNITUDE)

The open-loop transfer function $G(s)$ of a unity feedback control system is a complex quantity.

$$G(s) = x + jy \quad , \quad H(s) = 1 \qquad s = j\omega .$$

$$M = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s).H(s)}$$

$$\text{Magnitude} = M = \frac{x+jy}{1+x+jy} \quad \text{—— (1)}$$

Taking modulus

$$|M| = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}} \quad \text{—— (2)}$$

Squaring eqn (2) and Simplifying

$$M^2 = \frac{x^2+y^2}{(1+x)^2+y^2}$$

$$\Rightarrow \quad M^2\left[(1+x)^2+y^2\right] = x^2+y^2$$

$$\Rightarrow \quad M^2\left(1+x^2+2x+y^2\right) = x^2+y^2$$

$$\Rightarrow \quad M^2 + M^2x^2 + 2M^2x + M^2y^2 = x^2+y^2$$

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$$\Rightarrow (1-M^2)x^2 - 2M^2x + (1-M^2)y^2 = M^2.$$

$$\Rightarrow x^2 - \left(\frac{2M^2}{1-M^2}\right)x + y^2 = \frac{M^2}{1-M^2} \quad \text{--- (3)}$$

Making perfect square adding $\left(\frac{M^2}{1-M^2}\right)^2$ to both side to equ^n (3)

$$\Rightarrow x^2 - \frac{2M^2}{1-M^2}x + \left(\frac{M^2}{1-M^2}\right)^2 + y^2 = \frac{M^2}{1-M^2} + \left(\frac{M^2}{1-M^2}\right)^2$$

$$\Rightarrow \left(x - \frac{M^2}{1-M^2}\right)^2 + y^2 = \frac{M^2(1-M^2)+M^4}{(1-M^2)^2}$$

$$\Rightarrow \left(x - \frac{M^2}{1-M^2}\right)^2 + y^2 = \left(\frac{M}{1-M^2}\right)^2 \quad \text{--- (4)}$$

For different values of M, equ^n (4) represents a family of circles with centre at

$$\left(x = \frac{M^2}{1-M^2},\ y = 0\right).$$

and radius $\frac{M}{1-M^2}$

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For a particular circle the values of M (Magnitude of closed loop transfer function) is const, therefore these circles are called const. M-circle.

Fig. 7.11.1.

CONSTANT N-CIRCLE (PHASE ANGLES)

From eqn (1) the phase angle of the closed loop transfer function of a unity feedback control system is given by.

$$\phi = \angle \frac{C(s)}{R(s)} = \angle \frac{x+jy}{1+x+jy} \quad \text{—— (5)}$$

The phase angle ϕ

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right)$$

Taking tan on both sides.

$$\tan\phi = \tan\left(\tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right)\right)$$

$$= \frac{\tan\tan^{-1}\left(\frac{y}{x}\right) - \tan\tan^{-1}\frac{y}{1+x}}{1+\tan\tan^{-1}\frac{y}{x}\cdot\tan\tan^{-1}\left(\frac{y}{1+x}\right)}$$

$$= \frac{\frac{y}{x} - \frac{y}{1+x}}{1+\frac{y}{x}\times\frac{y}{1+x}}$$

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$$\tan\phi = \frac{\dfrac{y(1+x) - yx}{(x(1+x))}}{\dfrac{x(1+x) + y^2}{(x(1+x))}}$$

$$\tan\phi = \frac{y}{x^2+x+y^2}$$

put $\tan\phi = N$

$$N = \frac{y}{x^2+x+y^2}$$

$$\Rightarrow x^2+x+y^2 = \frac{y}{N}$$

$$\Rightarrow x^2+x+y^2 - \frac{y}{N} = 0$$

Making perfect square

$$x^2 + 2\cdot x\cdot\frac{1}{2} + \frac{1}{4} - \frac{1}{4} + y^2 - 2\cdot\frac{y}{2N} + \frac{1}{4N^2} - \frac{1}{4N^2} = \frac{1}{4} + \frac{1}{4N^2}$$

$$\Rightarrow \left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2N}\right)^2 = \left(\frac{1}{4}+\frac{1}{4N^2}\right) \quad (6)$$

for different values of N eqn (6) represent a family of circle with centre at $x = -\frac{1}{2}$, $y = \frac{1}{2N}$.

$$r = \sqrt{\frac{1}{4}+\frac{1}{4N^2}}$$

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M	Centre $x = \frac{M^2}{1-M^2}, y = 0$	Radius $r = \frac{M}{1-M^2}$
0.5	0.33	0.67
1.0	∞	∞^*
1.2	−3.27	2.73
1.6	−1.64	1.03
2.0	−1.33	0.67
3.0	−1.13	0.38

*Intersection with real axis at $x = -0.5$.

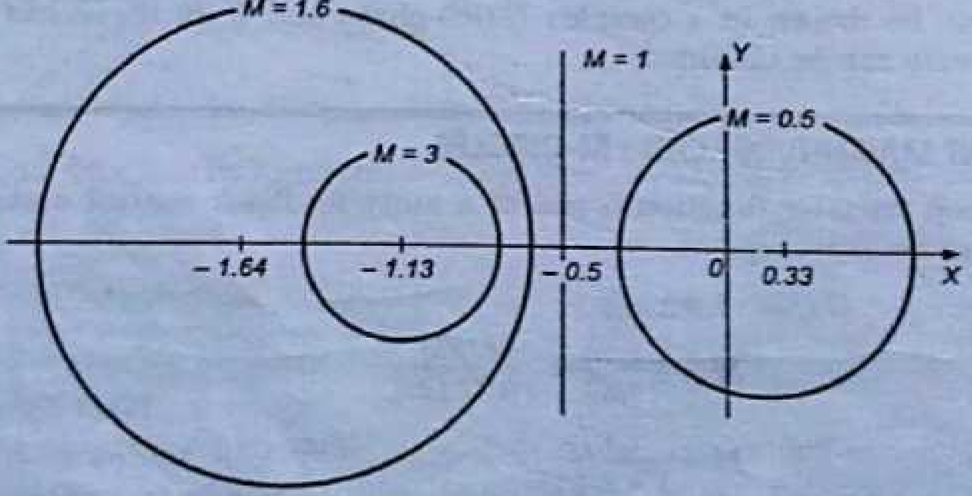

Fig. 7.11.1. M-circles.

In $G(j\omega)$ plane the Nyquist plot is superimposed on M-circle and the points of intersection give the magnitude of $C(j\omega)/R(j\omega)$ at different values of ω.

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ϕ	$N = \tan\phi$	Centre $x = -\frac{1}{2}, y = \frac{1}{2N}$	Radius $R = \sqrt{\frac{1}{4} + \frac{1}{4N^2}}$
− 90°	∞	0	0.5
− 60°	− 1.732	− 0.289	0.577
− 50°	− 1.19	− 0.42	0.656
− 30°	− 0.577	0.866	1.0
− 10°	− 0.176	− 2.84	2.88
0°	0	∞	∞
+ 10°	0.176	2.84	2.88
+ 30°	0.577	0.866	1.0
+ 50°	0.19	0.42	0.656
+ 60°	1.732	0.289	0.577
+ 90°	∞	0	0.5

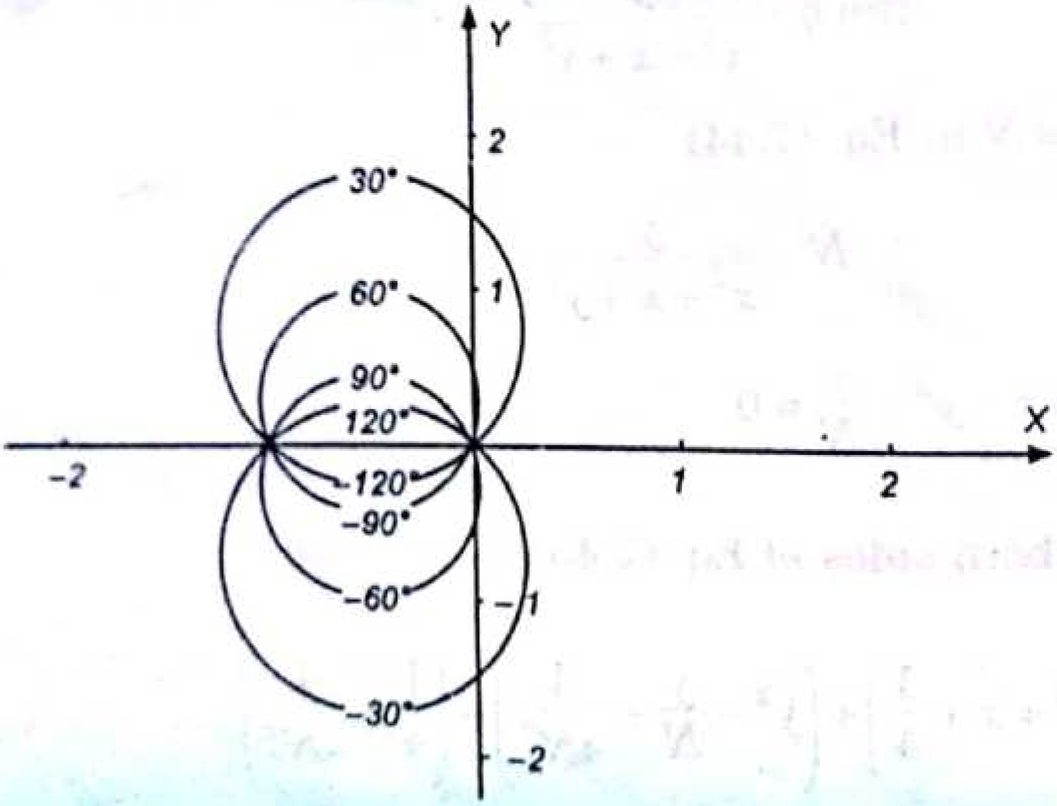

Fig. 7.12.1. *N*-circles.

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7.15. NICHOLS CHART

The constant M and constant N circles in $G(j\omega)$ plane can be used for the analysis and design of control systems. However the constant M and constant N circles in gain phase plane *i.e.* graph having gain in decibel along the ordinate and phase angle along the abscissa are prepared for system design and analysis as these plots supply information with less manipulations. The M and N circles of $G(j\omega)$ in the gain phase plane are transformed into M and N contours in rectangular co-ordinates. A point on the constant M loci in $G(j\omega)$ plane is transferred to the gain phase plane by drawing the vector directed from the origin of $G(j\omega)$ plane to the particular point on the M circle and then measuring the length db, angle in and degree. These values of length and angles are the coordinates of the corresponding point in the gain phase plane as shown in Fig. 7.15.1 and 7.15.2. The critical point in $G(j\omega)$, plane corresponds to the point of zero decibel and – 180° in the gain-phase plane. Plot of M and N circles in gain phase plane is shown in Fig. 7.15.3 and known as 'Nichols chart'.

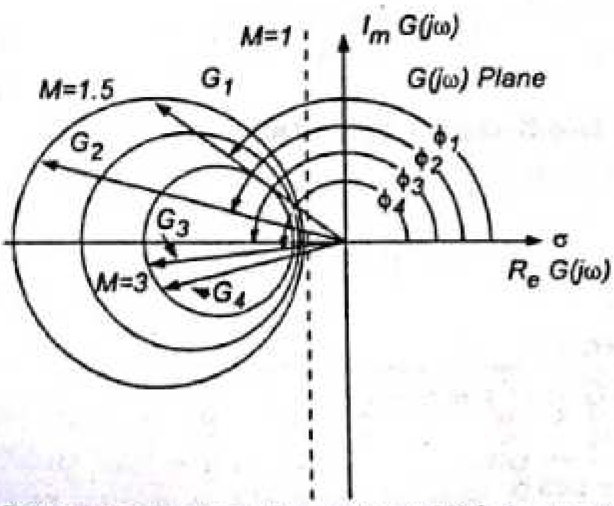

Fig. 7.15.1. *M*-circle transformation to Nichols chart.

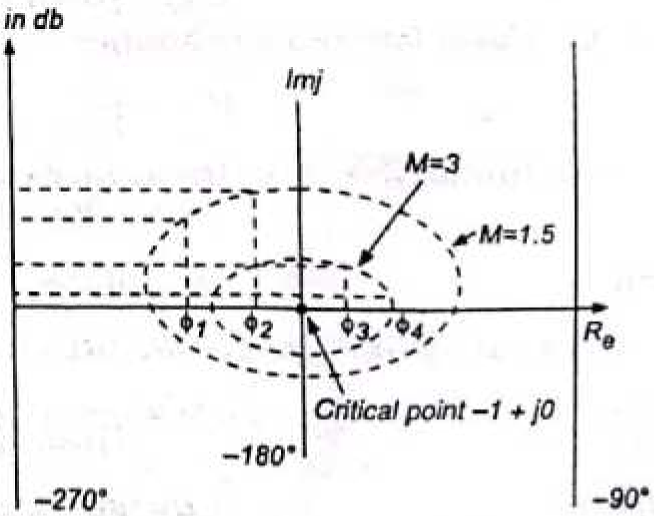

Fig. 7.15.2. *M*-circles in gain-phase plane.

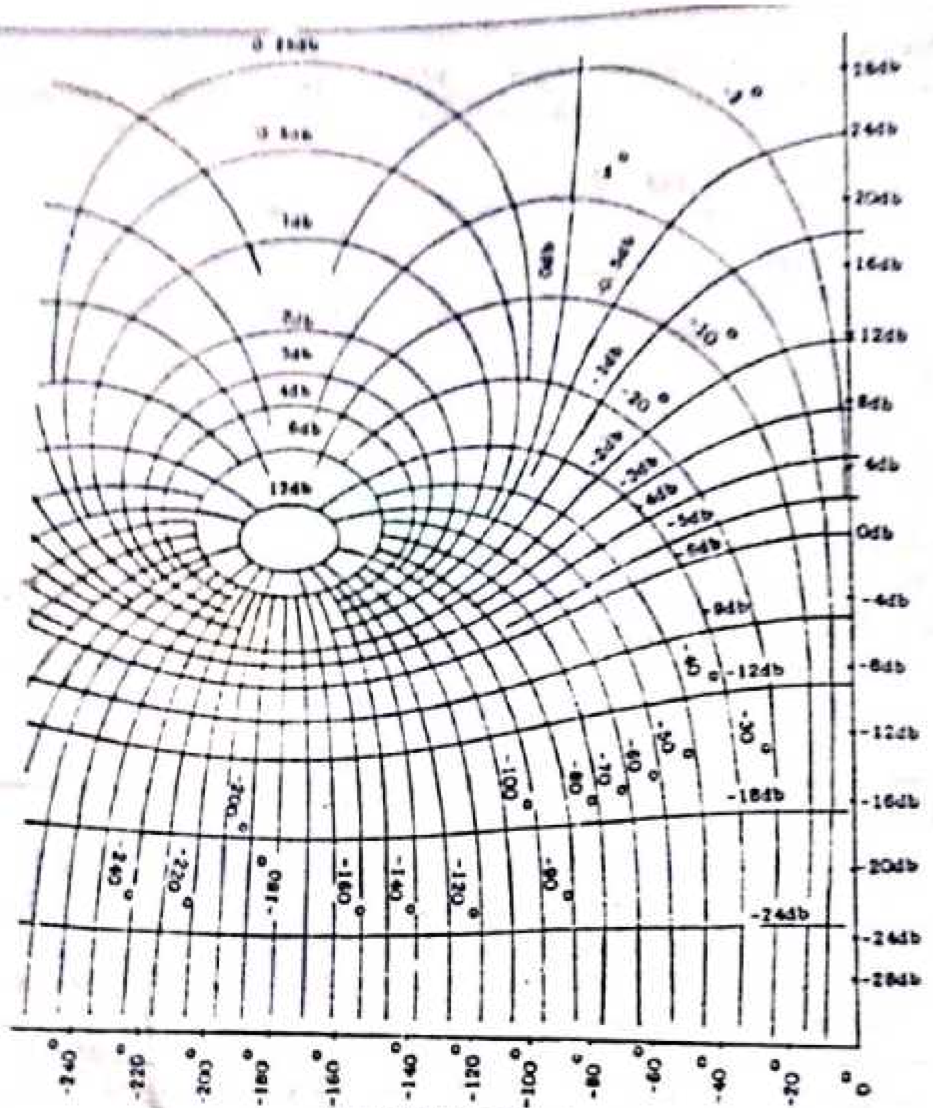

Fig. 7.15.3. Nichols chart.

Constant M and constant N circles in the Nichols chart are deformed into squashed circles. The complete Nichols chart extends for the phase angle of $G(j\omega)$ from 0 to $-360°$ but the region of $\angle Gj(\omega)$ generally used for analysis of systems is between $-90°$ and $-270°$. These curves repeat after every 180° interval. If the open-loop transfer function of the unity feedback system $G(s)$ is expressed as

$$G(s) = |G(s)|\, e^{j\theta} = |G(s)| [\cos\theta + j\sin\theta]$$

and the closed-loop transfer function

$$M(s) = \frac{G(s)}{1 + G(s)}$$

Substituting ($s = j\omega$) in the above equations the frequency functions are

$$G(j\omega) = |G(j\omega)| [\cos\theta + j\sin\theta]$$

and
$$M(j\omega) = Me^{j\phi} = \frac{G(j\omega)}{1 + G(j\omega)}$$

Eliminating $G(j\omega)$ from above two equations

$$M = \frac{|G(j\omega)|}{\sqrt{|G(j\omega)|^2 + 2|G(j\omega)|\cos\theta + 1}}$$

and
$$\phi = \tan^{-1}\frac{\sin\theta}{|G(j\omega)| + \cos\theta}$$

These equations define the plots in Nichols chart shown in Fig. 7.15.3.

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